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Maks A. Akivis
Vladislav V. Goldberg

Differential Geometry of Varieties with Degenerate Gauss Maps

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With 16 Figures

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Maks A. Akivis
Department of Mathematics
Jerusalem Institute of
Technology—Machon Lev
Jerusalem 91160
Israel
akivis@mail.jct.ac.il

Vladislav V. Goldberg
Department of Mathematical Sciences
New Jersey Institute of Technology
Newark, NJ 07102
USA
afs3@m.njit.edu

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Preface

1. The Gauss Map. The Gauss map of an oriented smooth surface X^2 in Euclidean space $\mathbb{E}^3$ is the mapping of X^2 into the unit sphere $S^2 \subset \mathbb{E}^3$:

$$\gamma : X^2 \to S^2,$$

by means of the family of the unit normals $\boldsymbol{n}$ to X^2. This map carries a point $x \in X^2$ to a point $p \in S^2$, where p is the terminal point of the vector $\boldsymbol{n}$ emanating from some fixed point $O \in \mathbb{E}^3$, $\gamma(x) = p$ (see Figure 0.1).

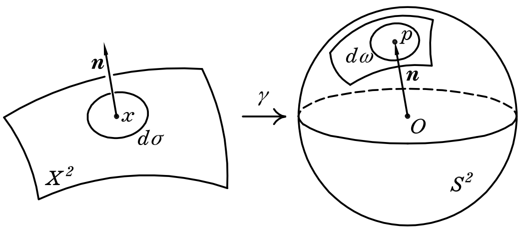

Figure 0.1

If $d\sigma$ is an area element of the surface X^2 and $d\omega$ is an area element of the spherical image of X^2, then

$$d\omega = K d\sigma,$$

where K is the Gaussian curvature of X^2 (see Gauss [Ga 27] or Stoker [Sto 61], p. 94).

The Gauss map γ is degenerate at a point $x \in X^2$ if $K = 0$ at this point, and the Gauss map γ is degenerate on the surface X^2 if the curvature K vanishes at all points of X^2. In this case the Gauss map γ maps the surface

X^2 into a curve $C \subset S^2$ (see Figure 0.2). The tangent planes to the surface X^2 depend on one parameter, and the surface X itself is an envelope of this family of tangent planes.

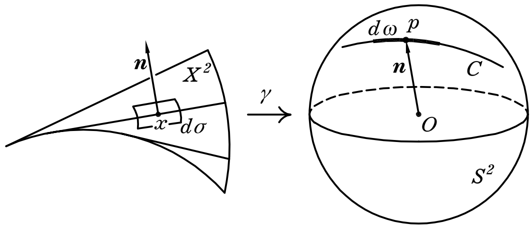

Figure 0.2

If the surface X^2 is defined in $\mathbb{E}^3$ by the equation $z = f(x, y)$, then the condition $K = 0$ is equivalent to the Monge–Ampère equation

$$rt - s^2 = 0,$$

where $r = z_{xx}, s = z_{xy}, t = z_{yy}$ (see Monge [Mon 50]). The surfaces with $K = 0$ are called *developable*. Such surfaces can be locally mapped isometrically into a plane. The latter property is the reason that surfaces with the vanishing Gaussian curvature are called developable: they can be "developed on the plane."

Developable surfaces are a well-known subject from the 19th century. Locally they are classified into three types: cones, cylinders, and torses (tangential developables). A torse is a one-parameter family of tangent lines to a fixed smooth space curve.

The definition of the Gauss map can be easily extended to a hypersurface $X = X^n$ of Euclidean space $\mathbb{E}^{n+1}$. The Gauss map of an oriented smooth hypersurface $X^n \subset \mathbb{E}^{n+1}$ is the mapping of V^n into the unit hypersphere $S^n \subset \mathbb{E}^{n+1}$:

$$\gamma : X \to S^n,$$

by means of the family of hypersurface normals $\boldsymbol{n}$. If $X \subset \mathbb{E}^{n+1}$ is given by the equation

$$z = f(x^1, \dots, x^n),$$

then the condition for its Gauss map to be degenerate has the form

$$\det(z_{ij}) = 0,$$

where $z_{ij} = \frac{\partial^2 z}{\partial x^i \partial x^j}$. If the submanifold X is of codimension $N - n > 1$, then the condition for its Gauss map γ to be degenerate has a more complicated form.

The fact that the Gauss map γ of $X \subset \mathbb{E}^{n+1}$ is degenerate is of projectively invariant nature. This is the reason that the degeneracy of the Gauss map can be defined in terms of projective differential geometry.

Let X be a smooth oriented submanifold of dimension n in the N-dimensional projective space $\mathbb{P}^N$, and let $\mathbb{G}(n, N)$ be the Grassmannian of n-dimensional subspaces of the space $\mathbb{P}^N$. Then the Gauss map γ of $X \subset \mathbb{P}^N$ is defined as the map

$$\gamma : X \to \mathbb{G}(n, N),$$

which carries a point $x \in X$ to the tangent subspace $T_x(X)$ to X at the point x, i.e.,

$$\gamma(x) = T_x(X).$$

The rank r of the map γ is called the *rank* of the submanifold X of dimension n. The rank r does not exceed n, and we assume that the rank r is constant on X.

In a projective space $\mathbb{P}^N$, a variety X of dimension n is said to be a *variety with a degenerate Gauss map* or a *tangentially degenerate variety* if the rank of its Gauss map $\gamma : X \to \mathbb{G}(n, N)$ is less than n. We use the term "variety" here instead of "submanifold" because X has a degenerate Gauss map, and hence it is differentiable almost everywhere (see Section 2.1) while a submanifold is differentiable everywhere.

In this book we study the geometry of varieties with degenerate Gauss maps, construct a classification of such varieties based on the structure of their focal images, and consider applications of the theory of such varieties to different problems of differential geometry and its applications.

Note that in higher dimensions the property through which developable surfaces can be mapped isometrically into a plane is not valid any longer. This is why we prefer to call a variety $X \subset \mathbb{P}^N$ for which $\operatorname{rank} \gamma < n$ a variety with a degenerate Gauss map or a tangentially degenerate variety. Note that some authors (Fisher, Ishikawa, Piontkowski, Mezzetti, Tommasi, Rogora, Wu, Zheng) call such varieties developable.

2. Developments in the Theory of Varieties with Degenerate Gauss Maps. As we mentioned earlier, the developable surfaces in the three-dimensional Euclidean space are a well-known subject from the 19th century. The torses (tangential developables) form a special class of ruled surfaces, namely developable ruled surfaces, and of necessity have singularities, at least along the original curve. There are numerous publications on developable surfaces. The main properties of developable surfaces can be found in most textbooks on differential geometry.

Mathematically developable surfaces are the subject of several branches of mathematics, especially of differential geometry and algebraic geometry. Recently developable surfaces have attracted attention through their relation with computer science (see, for example, the book by Pottmann and Wallner [PW 01]). They are widely used in industry, and are fundamental objects in computer-aided design (see for example, the paper by Hoschek and Pottmann [HoP 95]). Though singularities can be avoided in practical situations, the appearance of singularities in developable surfaces is essential to their nature. Thus the complete description of the structure of developable surfaces involves the singularity theory which was developed in the 20th century (see, for example, the books Bruce and Giblin [BG 92] and Porteous [Por 94]).

The multidimensional varieties X with degenerate Gauss maps of rank $r < n$ were considered by É. Cartan in [C 16] in connection with his study of metric deformation of hypersurfaces, and in [C 19] in connection with his study of manifolds of constant curvature. Yanenko [Ya 53] encountered these varieties in his study of metric deformation of submanifolds of arbitrary classes. Akivis [A 57, 62], Savelyev [Sa 57, 60], and Ryzhkov [Ry 60] systematically studied this kind of variety in a projective space $\mathbb{P}^N$. Brauner [Br 38], Wu [Wu 95], and Fischer and Wu [FW 95] studied such varieties in a Euclidean N-space $\mathbb{E}^N$. Akivis and Goldberg in their book [AG 93] investigated the multidimensional varieties with degenerate Gauss maps in Chapter 4.

Note that a relationship of the rank of varieties X and their deformation in a Euclidean N-space was indicated by Bianchi [Bi 05] who proved that a necessary condition for X to be deformable is the condition rank $X \leq 2$. Allendörfer [Al 39] introduced the notion of type t, $t = 0, 1, , \ldots, m = \dim X$, of X and proved that varieties X^{N-p}, $p > 1$, of type $t > 2$ in $\mathbb{E}^N$ are rigid. Note that both notions, the type and the rank, are projectively and metrically invariant, and that for a hypersurface, the type coincides with the rank.

Griffiths and Harris in their classical paper [GH 79] considered the varieties X with degenerate Gauss maps from the point of view of algebraic geometry. The paper [GH 79] was followed by Landsberg's paper [L 96] and book [L 99] and by the recently published book [FP 01] by Fischer and Piontkowski. The books [L 99] and [FP 01] have special sections devoted to varieties with degenerate Gauss maps. They are in some sense an update to the paper [GH 79]. In both books, following [GH 79], the authors employed a second fundamental form for studying developable varieties, gave detailed and more elementary proofs of some results in [GH 79], and reported on some recent progress in this area. In particular, in [FP 01] the authors gave a classification of developable varieties of rank two in codimension one.

In recent years many papers devoted to varieties with degenerate Gauss maps have appeared. Zak [Za 87] studied the Gauss maps of submanifolds of the projective space from the point of view of algebraic geometry. Ishikawa and Morimoto [IM 01] investigated the connection between such varieties and solutions of Monge–Ampère equations. Ishikawa [I 98, 99b] found real algebraic cubic nonsingular hypersurfaces with degenerate Gauss maps in $\mathbb{RP}^N$ for $N = 4, 7, 13, 25$, and in [I 99a] he studied singularities of C^∞-hypersurfaces with degenerate Gauss maps. Rogora [Rog 97] and Mezzetti and Tommasi [MT 02a, 02c] also considered varieties with degenerate Gauss maps from the point of view of algebraic geometry. Piontkowski [Pio 01, 02a, 02b] considered in $\mathbb{P}^N$ complete varieties with degenerate Gauss maps with rank equal to two, three, and four and with all singularities located on a hyperplane at infinity. The reader can find more details on all these results in the Notes to Chapter 2.

The contents of this book are connected with the theory of singularities of differentiable mappings. There are numerous publications on this topic. In particular, in the book [AVGL 89] by Arnol'd, Vasil'ev, Goryunov, and Lyashko, which is devoted to investigations of singularities of differentiable mappings, their classification, and their applications, the authors consider the singularities of the Grassmann mappings of submanifolds of the Euclidean space and the projective space. Many papers (for example, [Sh 82] by Shcherbak and [I 00b] by Ishikawa) are devoted to a classification of isolated singular points of curves in the Euclidean space and the projective space.

As a rule, the singular points we consider on a variety with a degenerate Gauss map are not isolated (see Section 2.4).

We outline here what distinguishes our book on varieties with degenerate Gauss maps from other literature on this subject:

i) In the current book the authors systematically study the differential geometry of varieties with degenerate Gauss maps. They apply the main methods of differential geometry: the tensor analysis, the method of exterior forms, and the moving frame method.

ii) Western authors were not familiar with the results obtained by Russian geometers in the 1960s (Akivis, Ryzhkov, Savelyev). Some of the results presented by western geometers had been known for years. We present all these results in their historical perspective.

iii) In the study of varieties with degenerate Gauss maps, the authors *systematically* use the focal images (the focal hypersurfaces and the focal hypercones) associated with such varieties. These images were first introduced by Akivis in [A 57]. They allow the authors to describe the geometry of the varieties with degenerate Gauss maps and give their classification. Note that in algebraic geometry, the focal hypersurfaces are called the discriminant varieties.

iv) In the complex projective space, every plane generator L of a variety with a degenerate Gauss map carries singular points. The question is whether these singular points should be included in L. Our point of view is that it is very useful to include them in L; this simplifies the exposition. Many algebraic geometers who study this subject do not consider singular points as a part of L, and this makes their exposition of the results more complicated.

Note also that in most of the books and papers where the singularities of differentiable mappings are considered, the authors investigate only isolated singularities. But the singularities of Gauss maps comprise algebraic curves or hypersurfaces in the plane generators of varieties with degenerate Gauss maps.

v) It appeared that the Griffiths–Harris conjecture on the structure of varieties with degenerate Gauss maps is not complete. As we show in this book (see also our paper [AG 01a] and the paper [AGL 01] by Akivis, Goldberg, and Landsberg), the basic types of varieties with degenerate Gauss maps include not only cones and torses but also hypersurfaces with degenerate Gauss maps. Note that such hypersurfaces form a very wide class of varieties with degenerate Gauss maps.

vi) When the authors were writing this book, they found some new results on the varieties with degenerate Gauss maps. Some of them were already published and some are in papers submitted for publication. Among these results are a new classification of such varieties (see Akivis and Goldberg [AG 01a]), a detailed investigation of Sacksteder–Bourgain hypersurfaces (see Akivis and Goldberg [AG 01b]), finding an affine analogue of the Hartman–Nirenberg cylinder theorem (see [AG 02a]), establishing the relation between the smooth lines on projective planes over two-dimensional algebras and the varieties with degenerate Gauss maps (see Akivis and Goldberg [AG 02b]), application of the duality principle for construction of varieties with degenerate Gauss maps (see Akivis and Goldberg [AG 02b]), and a description of a new class varieties with degenerate Gauss maps (twisted cones) (see Akivis and Goldberg [AG 03b]).

vii) In this book we consider a very large number of examples. Some of these examples (such as the twisted cones and some algebraic hypersurfaces in $\mathbb{P}^4$) are considered here for the first time, and other examples (such as the cubic symmetroid in $\mathbb{P}^5$ and its projection onto $\mathbb{P}^4$) were known earlier but are considered here from a new point of view.

viii) The authors give a new definition for the dual defect of a variety with a degenerate Gauss map and for dually degenerate varieties with degenerate Gauss maps (see p. 72). This new definition is better than the usual definition of the dual defect given on p. 71: while by old definition all varieties with degenerate Gauss maps are dually degenerate, by the new definition, they can be both dually degenerate and dually nondegenerate. Moreover, while by the old definition, the dual defect δ_* of a dually nondegenerate variety with degenerate Gauss map equals its Gauss defect, $\delta_* = \delta_\gamma > 0$, by the new definition, the dual defect δ_* of such a variety equals 0, $\delta_* = 0$, and this is more appropriate for a *dually nondegenerate* variety.

In addition to varieties with degenerate Gauss maps, algebraic geometry studies other kinds of degenerate varieties (such as secantly degenerate and dually degenerate varieties; see, for example, the paper [GH 78] by Griffiths and Harris; the books [L 99] by Landsberg, pp. 4, 16, and 52; [T 01] by Tevelev, Chapters 6, 9; and [Ha 92] by Harris, pp. 197–199). Not as many secantly degenerate, dually degenerate, and degenerate varieties of other kinds are known. For example, there is only one secantly degenerate variety in the projective space $\mathbb{P}^5$, namely, the Veronese variety (see Sasaki [Sas 91] and Akivis [A 92]). In this connection, note also that all smooth dually degenerate varieties of dimension $n \leq 10$ are listed (see for example, the book [T 01] by Tevelev, Chapter 9, or Notes to Section 2.5 of this book where the appropriate references are given).

Unlike the classes of these degenerate varieties, the varieties with degenerate Gauss maps compose a much wider class. In particular, the arbitrariness of the class of torsal varieties is equal to some number of functions of two variables, and the arbitrariness of the class of hypersurfaces with degenerate Gauss maps of rank r in the space $\mathbb{P}^N$ (as well as their dual image, smooth tangentially nondegenerate subvarieties, for which $r = n$) is equal to $N - r$ functions of r variables. Hence, the study of the varieties with degenerate Gauss maps in the space $\mathbb{P}^N$ is of considerable interest.

Note that in the book only dually nondegenerate varieties with degenerate Gauss maps are under investigation. For such varieties, the system of second fundamental forms always contains at least one nondegenerate form of rank r, and for them not only the focus hypersurfaces but also the focus hypercones whose vertices are the tangent subspaces of the variety X are correctly defined.

3. The Contents of the Book. The book consists of five chapters. In Chapter 1, we give the basic notions and results of vector spaces and projective space, consider the main topics associated with differentiable manifolds, and study some algebraic varieties, namely, Grassmannians and determinant submanifolds.

In Chapter 2, we introduce the basic notions associated with a variety in a projective space $\mathbb{P}^N$, define the rank of a variety and varieties with degenerate Gauss maps, present the main examples of varieties with degenerate Gauss maps (cones, torses, hypersurfaces, joins, etc.), study the duality principle and its applications, consider another example of submanifolds with degenerate Gauss maps (the cubic symmetroid) and correlative transformations, and investigate a hypersurface with a degenerate Gauss map associated with a Veronese variety and find its singular points. The reader can find more details on Chapters 1 and 2 in the Contents.

In Chapter 3, we define the Monge–Ampère foliation associated with a variety with a degenerate Gauss map of dimension n, derive the basic equations of varieties with degenerate Gauss maps, prove a characteristic property of such varieties (the leaves of the Monge–Ampère foliation are flat), and consider focal images of such varieties (the focus hypersurfaces and the focus hypercones). In this chapter we also study varieties with degenerate Gauss maps not only in the complex projective space but also in the real projective space, the affine space, the Euclidean space, and the non-Euclidean spaces. We prove that in these spaces there are varieties with degenerate Gauss maps without singularities, and we introduce and investigate an important class of varieties with degenerate Gauss maps without singularities, the so-called the Sacksteder–Bourgain hypersurface. Note that Sacksteder and Bourgain constructed examples of hypersurfaces with degenerate Gauss maps in the affine space $\mathbb{A}^4$. In Section 3.4 (see also the paper by Akivis and Goldberg [AG 01b]), we prove that the hypersurfaces constructed by them are locally equivalent, and we construct a series of hypersurfaces with degenerate Gauss maps in the affine space $\mathbb{A}^N$ generalizing the Sacksteder–Bourgain hypersurface.

In Chapter 4, in the projective space $\mathbb{P}^N$, we consider the basic types of varieties with degenerate Gauss maps: cones, torsal varieties, hypersurfaces with degenerate Gauss maps. For each of these types, we consider the structure of their focal images and find sufficient conditions for a variety to belong to one of these types (for torsal varieties our condition is also necessary). The classification of varieties X with degenerate Gauss maps presented in this chapter is based on the structure of the focal images of X. In a series of theorems, we establish this connection. We prove that varieties with degenerate Gauss maps that do not belong to one of the basic types are foliated into varieties of basic types. Finally, we prove an embedding theorem for varieties with degenerate Gauss maps and find sufficient conditions for such a variety to be a cone. In this chapter, we also consider varieties with degenerate Gauss maps in the affine space $\mathbb{A}^N$ and find a new affine analogue of the Hartman–Nirenberg cylinder theorem. We consider here parabolic hypersurfaces in the space $\mathbb{P}^4$ (i.e., the hypersurfaces X with degenerate Gauss maps of rank $r = 2$ that have a double focus F on each rectilinear generator L). We also prove existence theorems for some varieties with degenerate Gauss maps, for example, for twisted cones in $\mathbb{P}^4$ and $\mathbb{A}^4$, and we establish a structure of twisted cones in $\mathbb{P}^4$. This structure allows us to find a procedure for construction of twisted cylinders in $\mathbb{A}^4$.

Chapter 5 is devoted to further examples and applications of the theory of varieties with degenerate Gauss maps. As the first application, we prove that lightlike hypersurfaces in the de Sitter space $\mathbb{S}_1^{n+1}$ have degenerate Gauss maps, that their rank $r \leq n-1$, and that there are singular points and singular submanifolds on them. We classify singular points and describe the structure of lightlike hypersurfaces carrying singular points of different types. Moreover, we establish the connection of this classification with that of canal hypersurfaces of the conformal space. As the second application, we establish a relation of the theory of varieties with degenerate Gauss maps in projective spaces with the theory of congruences and pseudocongruences of subspaces and show how these two theories can be applied to the construction of induced connections on submanifolds of projective spaces and other spaces endowed with a projective structure. As the third application, we consider smooth lines on projective planes over the complete matrix algebra $\mathbb{M}$ of order two, the algebra $\mathbb{C}$ of complex numbers, the algebra $\mathbb{C}^1$ of double numbers, and the algebra $\mathbb{C}^0$ of dual numbers. For the algebras, $\mathbb{C}, \mathbb{C}^1$, and $\mathbb{C}^0$, in the space $\mathbb{RP}^5$, to these smooth lines there correspond families of straight lines describing three-dimensional point varieties X^3 with degenerate Gauss maps of rank $r \leq 2$. We prove that they represent examples of different types of varieties X^3 with degenerate Gauss maps.

Sections, formulas, and figures in the book are numbered within each chapter. Each chapter is accompanied by notes containing remarks of historical and bibliographical nature and some supplementary results pertinent to the main content of the book. A fairly complete bibliography on multidimensional varieties with degenerate Gauss maps, a list of notations, an author index, and a subject index are at the end of the book.

Bibliographic references give the author's last name followed by the first letter(s) of the author's last name and the last two digits of the year in square brackets, for example, Blaschke [Bl 21]. Note that in the bibliography, in addition to the original article being cited, reviews of the article in major mathematical review journals (*Jahrbuch für Fortschritte der Mathematik*, *Zentralblatt für Mathematik*, *Mathematical Reviews*) are referenced.

4. General Remarks for the Reader. This book is intended for graduate students whose field is differential geometry, as well as for mathematicians and teachers conducting research in this subject. It can be used in special graduate courses in mathematics.

In our presentation we use the tensorial methods in combination with the methods of exterior differential forms and moving frames of Élie Cartan. The reader is assumed to be familiar with these methods, as well as with the basics of modern differential geometry. However, in Chapter 1 we recall basic facts of tensor calculus and the method of moving frame in the form in which they will be used in the book. Many other concepts of differential geometry are explained briefly in the text; some are given without explanation. As references, the books [KN 63] by Kobayashi and Nomizu, [BCGGG 91] by Bryant et al., and [C 45] by É. Cartan are recommended. In the book [Sto 69] by Stoker, the reader can find the main notions and theorems of elementary differential geometry that are necessary for reading this book. We also recommend our book Akivis and Goldberg [AG 93], in which the projective differential geometry of general submanifolds and some of their most important special classes were developed systematically. We will often refer to this book.

All functions, vector and tensor fields, and differential forms are assumed to be differentiable almost everywhere. As a rule, we use the index notations in our presentation. We believe this allows us to obtain a deeper understanding of the essence of problems in local differential geometry.

Note also that if we impose a restriction on a variety, then, as a rule, we assume that this condition holds at all points of this variety. More precisely, we consider only the domain of the variety where this restriction holds.

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Jerusalem, Israel *Maks A. Akivis*

Newark, New Jersey, USA *Vladislav V. Goldberg*

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Chapter 1

Foundational Material

In Section 1.1, we discuss the basic notions and results of vector spaces, vectors and tensors in them, and the general linear group. In Section 1.2, we consider the main topics associated with differentiable manifolds: tangent spaces, frame bundles, mappings, exterior differential calculus, Cartan's lemma, completely integrable systems, the Frobenius theorem, Cartan's test for a system in involution, the structure equations of a differentiable manifold and of the general linear group, and affine connections. Section 1.3 is dedicated to a projective space—we consider projective transformations, projective frames, and the structure equations of a projective space, the duality principle, projectivization, classical homogeneous spaces (affine, Euclidean, non-Euclidean), and their transformations. In Section 1.4 we demonstrate the geometric and analytic methods of specialization of moving frames by considering the geometry of a curve in the projective plane. Finally, in Section 1.5, we study some algebraic varieties, namely, Grassmannians and determinant submanifolds (Segre and Veronese varieties).

1.1 Vector Space

1.1.1 The General Linear Group. In what follows, the notion of a finite-dimensional vector space L^n over the field of real or complex numbers will play an important role. We will not state here the basic axioms and properties of a vector space—they can be found in any textbook on linear algebra. Note only that a *frame* (or a *basis*) of an n-dimensional vector space L^n is a system consisting of n linearly independent vectors $e_1, e_2, \ldots, e_n$. A transition from one frame $R = \{e_1, e_2, \ldots, e_n\}$ to another frame $R' = \{e'_1, e'_2, \ldots, e'_n\}$ is determined by the relation

$$e'_i = e_j \cdot a^j_i, \quad i, j = 1, \ldots, n, \tag{1.1}$$

where $a = (a^j_i) \in \mathbf{GL}(n)$ is a nonsingular square matrix. (In these formulas and everywhere in the sequel the Einstein summation convention is used, i.e., it is assumed that summation is carried out over the indices that appear twice: once above and once below.) Thus, the family $\mathcal{R}(L^n)$ of frames in the space L^n depends on n^2 parameters.

Let us fix a frame R_0, and let R_a be an arbitrary frame in the space L^n, where a is a set of parameters determining the location of the frame R_a with respect to the frame R_0:

$$R_a = R_0 \cdot a. \tag{1.2}$$

In (1.2) the entries of the matrix $a = (a_i^j)$ are global functions on the manifold $\mathcal{R}(L^n)$. Equation (1.2) shows that the frame R_a is a differentiable function (in fact, linear) of parameters $a = (a_i^j)$. Let R_{a+da} be a frame near the frame R_a on the manifold $\mathcal{R}(L^n)$. Then the transition from the frame R_a to the frame R_{a+da} is described as follows:

$$\begin{aligned} R_{a+da} &= R_0 \cdot (a + da) = R_a \cdot (a^{-1}) \cdot (a + da) \\ &= R_a \cdot (I + a^{-1} \cdot da) = R_a \cdot (I + \omega_a), \end{aligned} \tag{1.3}$$

where

$$\omega_a = a^{-1} \cdot da \tag{1.4}$$

is a differential 1-form. Now (1.3) implies that

$$dR_a = R_a \cdot \omega_a. \tag{1.5}$$

The form ω_a is a left-invariant form with respect to the transformations of the group $\mathbf{GL}(n)$. In fact, let g be a fixed element from the group $\mathbf{GL}(n)$. Then

$$\begin{aligned} \omega_{ga} &= (ga)^{-1} d(ga) \\ &= (a^{-1} \cdot g^{-1} \cdot (dg \cdot a + g \cdot da) \\ &= a^{-1}(g^{-1} \cdot g) da = a^{-1} \cdot da = \omega_a \end{aligned} \tag{1.6}$$

(because $dg = 0$). The fact that the forms ω_a defined by (1.4) are left-invariant guarantees that all our constructions do not depend on the choice of the initial frame R_0 in L^n.

Because $R_a = \{e_1, e_2, \dots, e_n\}$, by setting $\omega_a = (\omega_i^j)$, we can write equations (1.5) in the vector form

$$de_i = e_j \cdot \omega_i^j. \tag{1.7}$$

Following É. Cartan, we shall write equations (1.7) in the form

$$de_i = \omega_i^j \cdot e_j. \tag{1.8}$$

1.1.2 Vectors and Tensors. Let us find the law of transformation of the coordinates of a vector under transformations of a frame in the space L^n. Suppose we have two frames R and R' whose vectors are connected by relations (1.1). An arbitrary vector x can be represented in the form of linear combinations of the vectors of these two frames:

$$x = x^i e_i = {}'x^i e'_i. \tag{1.9}$$

Using formulas (1.1), we find from (1.9) that

$$x^i = a^i_j \cdot {}'x^j, \;\; {}'x^i = \widetilde{a}^i_j x^j, \tag{1.10}$$

where $(\widetilde{a}^i_j)$ is the inverse matrix of the matrix (a^i_j).

In what follows, it will be more convenient for us to replace equations (1.10) with equivalent differential equations. We assume that the vector x is unchanged under transformations of a frame, i.e., we assume that $dx = 0$. If we differentiate the first equation of (1.9) and apply formulas (1.8), we obtain

$$0 = dx^i e_i + x^i de_i = (dx^i + x^j \omega^i_j) e_i.$$

The linear independence of the vectors e_i implies that

$$dx^i + x^j \omega^i_j = 0. \tag{1.11}$$

Equations (1.11) are the desired differential equations, which are equivalent to equations (1.10). Equations (1.10) can be recovered by integrating equations (1.11).

A *covector* is a linear function $\xi(x)$ of the vector variable x. The coordinate representation of $\xi(x)$ is $\xi(x) = \xi_i x^i$. Because this expression does not depend on the choice of the frame R, we have $\xi_i x^i = \text{const}$. The linear forms $\xi(x)$ in L^n form the vector space $(L^n)^*$ called the *dual space* of the space L^n.

Next, let us find the differential equations for coordinates of a covector ξ_i. It follows from the definition of a covector that its contraction $\xi_i x^i$ with coordinates x^i of an arbitrary vector x is constant, i.e., this contraction does not depend on the choice of frame:

$$\xi_i x^i = \text{const}.$$

Differentiating this relation and using formulas (1.11), we find that

$$d\xi_i x^i + \xi_i dx^i = (d\xi_i - \xi_j \omega^j_i) x^i = 0.$$

Because this holds for any vector x^i, it follows that

$$d\xi_i - \xi_j \omega_i^j = 0. \tag{1.12}$$

Similar equations can be derived for a tensor of any type. For example, let us consider a tensor t of the type (1, 2) with components t^i_{jk}. It follows from the definition of such a tensor that its contraction with coordinates x^j, y^k, and ξ_i of arbitrary vectors x, y and an arbitrary covector ξ does not depend on the choice of frame:

$$t^i_{jk} x^j y^k \xi_i = \text{const}.$$

Differentiating this relation and using formulas (1.11) and (1.12), we find the differential equations that the components t^i_{jk} of the tensor t satisfy:

$$dt^i_{jk} - t^i_{lk}\omega^l_j - t^i_{jl}\omega^l_k + t^l_{jk}\omega^i_l = 0. \tag{1.13}$$

By integrating equations (1.12) and (1.13), we can get the laws of transformation of the coordinates of a covector ξ_i and the tensor t^i_{jk} under transformation (1.11) of a frame:

$$\begin{aligned} {}'\xi_i &= a^j_i \xi_j, \\ {}'t^i_{jk} &= a^l_j a^m_k \widetilde{a}^i_p t^p_{lm}. \end{aligned} \tag{1.14}$$

To simplify the form of equations (1.11), (1.12), (1.13), and similar equations, it is convenient to introduce the differential operator ∇ defined by the following formulas:

$$\begin{aligned} \nabla x^i &= dx^i + x^j \omega^i_j, \\ \nabla \xi_i &= d\xi_i - \xi^j \omega^i_j. \end{aligned} \tag{1.15}$$

Using this operator, we can write equations (1.11), (1.12), and (1.13) in the form

$$\nabla x^i = 0, \quad \nabla \xi_i = 0, \quad \nabla t^i_{jk} = 0. \tag{1.16}$$

In addition to the vectors and tensors considered above that were invariant under transformations of a frame, we will encounter objects that get multiplied by some number under transformations of a frame. This number depends on the choice of basis and some other factors. Such objects are called *relative vectors* and *relative tensors*. Their coordinates satisfy equations that are slightly different from equations (1.16). For example, for a relative tensor of type (1, 2), these equations have the form

$$\nabla t^i_{jk} = \theta t^i_{jk}, \tag{1.17}$$

where θ is a closed linear differential form. The following law of transformation:

$$'t^i_{jk} = \lambda a^l_j a^m_k \widetilde{a}^i_p t^p_{lm} \tag{1.18}$$

corresponds to equations (1.17). If we differentiate (1.17), we obtain (1.18), where $\theta = d\lambda$.

The simplest tensor is the tensor of type (0, 0) or an *absolute invariant*, i.e., a quantity K that does not depend on the choice of frame. For this quantity, equation (1.16) becomes

$$dK = 0. \tag{1.19}$$

A *relative invariant* is a quantity K that is multiplied by a scalar under transformations of a frame. For this quantity, equation (1.17) becomes

$$dK = \theta K. \tag{1.20}$$

1.2 Differentiable Manifolds

1.2.1 The Tangent Space, the Frame Bundle, and Tensor Fields. The second basic that is needed is the notion of a *differentiable manifold.* We give only the main points of the definition here; for more detail, we refer the reader to other books (see, for example, the books [KN 63] by Kobayashi and Nomizu, [Di 70, 71] by Dieudonné, or [Va 01] by Vasil'ev).

A neighborhood of any point of a differentiable manifold M is homeomorphic to an open simply connected domain of the coordinate space $\mathbb{R}^n$ (or $\mathbb{C}^n$ if the manifold M is complex). This allows us to introduce coordinates in the neighborhood of any point of the manifold. The number n is the *dimension* of the manifold M.

If neighborhoods of two points of the manifold M have a nonempty intersection, then the two coordinate systems defined in this intersection are connected by means of invertible differentiable functions. The differentiability class of these functions is called the *class* of the differentiable manifold. Coordinates defined in a neighborhood of a point of a differentiable manifold admit invertible transformations of the same class of differentiability. In what follows, we will assume the differentiable manifolds under considerations to be of class C^∞, and in the complex case we will assume them to be analytic.

Consider an n-dimensional differentiable manifold M^n and a point $x \in M^n$. In a neighborhood of the point x, we introduce coordinates in such a way that the point x itself has zero coordinates. Let $x^i = x^i(t)$ be a smooth curve passing through the point x. We parameterize this curve so that $x^i(0) = 0$.

The quantities $\left.\frac{dx^i}{dt}\right|_{t=0} = \xi^i$ are called the *coordinates of the tangent vector* ξ to the curve under consideration at the point x. The parametric equations of the curve can be written as $x^i(t) = \xi^i t + o^i(t)$, where $o^i(t)$ are infinitesimals of orders higher than t.

The set of tangent vectors to all curves passing through a point $x \in M^n$ forms an n-dimensional vector space. This space is called the *tangent space* to the manifold M^n at the point x and is denoted by $T_x(M^n)$. The set of all tangent spaces TM^n of the manifold M^n along with their natural projections $TM^n \to M^n$ is called its *tangent bundle* and is denoted by $T(M^n)$. An element of the tangent bundle is a pair (x, ξ), where $x \in M^n$ and $\xi \in T_x(M^n)$. This explains why the tangent bundle is also a differentiable manifold of dimension $2n$, $\dim T(M^n) = 2n$.

Next, we consider the set of all possible frames $R_x = \{e_i\}$ in each tangent space. This set can be viewed as a fiber of a fibration $\mathcal{R}(M^n)$ called the *frame bundle* over the manifold M^n. Because the family of frames at a fixed point x depends on n^2 parameters, the dimension of the frame bundle $\mathcal{R}(M^n)$ is equal to $n + n^2$: $\dim \mathcal{R}(M^n) = n + n^2$.

Let ξ be a vector of the space $T_x(M^n) : \ \xi \in T_x(M^n)$. The decomposition of this vector relative to the basis $\{e_i\}$ has the form

$$\xi = \omega^i(\xi) e_i, \tag{1.21}$$

where $\omega^i(\xi)$ are the coordinates of the vector ξ with respect to the basis $\{e_i\}$. These coordinates are linear forms constituting a *cobasis* (a dual basis) in the space $T_x(M^n)$. This cobasis is a basis in the dual space $T_x^*(M^n)$. An element of the dual space is a linear form over $T_x(M^n)$. It follows from formula (1.21) that

$$\omega^i(e_j) = \delta_j^i. \tag{1.22}$$

The set of spaces $T_x^*(M^n)$ forms the *cotangent bundle* $T^*(M^n)$ over the manifold M^n.

Because every tangent space $T_x(M^n)$ is an n-dimensional vector space, we can consider tensors of different types in this space. A *tensor field* $t(x)$ is a function that assigns to each point $x \in M^n$ the value of the tensor t at this point. We will assume that the function $t(x)$ is differentiable as many times as we need.

In each space $T_x(M^n)$, the frames $\{e_i\}$ admit transformations whose differentials can be written in the form (1.8). Because later we will also consider displacements of the point x along the manifold M^n, we rewrite formulas (1.8) in the form

$$\delta e_i = \pi_i^j e_j, \tag{1.23}$$

where δ denotes differentiation when the point x is held fixed, i.e., δ is the restriction of the operator of differentiation d to the fiber $R_x(M^n)$ of the frame bundle $\mathcal{R}(M^n)$, and the forms π_i^j are invariant forms of the general linear group $\mathbf{GL}(n)$ of frame transformations in the space $T_x(M^n)$. Parameters defining the location of a frame in the space $T_x(M^n)$ are called *secondary* (or *fiber*) *parameters*, in contrast to *principal parameters*, which define the location of the point x in the manifold M^n. This is why the symbol δ is called the *operator of differentiation with respect to the secondary parameters* and the 1-forms π_j^i are called the *secondary* (or *fiber*) *forms*.

If a tensor field is given on the manifold M^n, then the coordinates of this field must satisfy equations of type (1.13) at any point of this field. For example, the coordinates t_{jk}^i of the tensor field $t(x)$ of type $(1,2)$ depend not only on a point x but also on the frame R_x attached to the point x, so that $t_{jk}^i = t_{jk}^i(x, R_x)$. If the point x is held fixed, then this dependence can be written in the form of the following differential equations:

$$\delta t_{jk}^i - t_{lk}^i \pi_j^l - t_{jl}^i \pi_k^l + t_{jk}^l \pi_l^i = 0. \tag{1.24}$$

If, in accordance with formulas (1.15), we denote the left-hand side of this equation by $\nabla_\delta t_{jk}^i$, then this equation takes the form

$$\nabla_\delta t_{jk}^i = 0. \tag{1.25}$$

If the point x moves along the manifold M, then for a tensor field $t_{jk}^i(x)$, equations (1.24) and (1.25) have the form

$$dt_{jk}^i - t_{lk}^i \omega_j^l - t_{jl}^i \omega_k^l + t_{jk}^l \omega_l^i = t_{jkl}^i \omega^l \tag{1.26}$$

and

$$\nabla t_{jk}^i = t_{jkl}^i \omega^l, \tag{1.27}$$

where ω^l are basis forms of the manifold M^n.

1.2.2 Mappings of Differentiable Manifolds. Let M and N be two manifolds of dimension m and n, respectively, and let $f : M \to N$ be a differentiable mapping of M into N. Consider a point $a \in M$, its image $b = f(a) \in N$ under the mapping f, and coordinate neighborhoods U_a and U_b of the points a and b. The mapping f defines a correspondence

$$y^u = f^u(x^i), \;\; i = 1, \dots, m, \; u = 1, \dots, n,$$

between coordinates of points $x \in U_a$ and $y \in U_b$. A mapping f is differentiable of class p, $f \in C^p$, if and only if the functions f^u are differentiable scalar functions of the same class p. If the functions f^u are infinitely differentiable functions, then the mapping f belongs to class C^∞, and if the functions f^u are analytic functions, then $f \in C^\omega$.

Consider the matrix

$$\mathcal{M} = \left(\frac{\partial y^u}{\partial x^i}\right)$$

having n rows and m columns. This matrix is called the *Jacobi matrix* of the mapping f. It is obvious that the rank r of this matrix satisfies the condition

$$r \leq \min(m, n).$$

It is also obvious that the rank r depends on a point $x \in U_a$. If the rank reaches its maximal value at a point x, i.e., $r = \min(m, n)$, then the mapping f is said to be *nondegenerate* at the point x, and the point x itself is called a *regular* point of a mapping f. If $r < \min(m, n)$ at a point x, then the point x is called a *singular* point of the mapping f.

The following relations can exist between the dimensions m and n:

a) $m < n$. In this case, in a neighborhood U_a of a regular point a, a mapping f is *injective*. If $a \in M$ is a regular point of the mapping f, then $b = f(a)$ is a regular point of the submanifold $V = f(M) \subset N$, and the map f carries a sufficiently small neighborhood of the point a into a spherical neighborhood of the point $b = f(a)$. Moreover, the tangent subspace $T_b(V)$ at a regular point b is an m-dimensional subspace of the tangent subspace $T_b(N)$ whose dimension is equal to n. In particular, if $m = 1$, the submanifold V is a *curve* in N, and if $m = n - 1$, the submanifold V is a *hypersurface* in N.

b) $m > n$. In this case, in a neighborhood U_a of a regular point a, a mapping f is *surjective*. In U_a, this mapping defines a foliation whose leaves F_y are the complete preimages $f^{-1}(y)$ of the points $y \in U_b$, where $b = f(a)$. The dimension of a leaf is equal to $m - n$, and the dimension of the subspace tangent to the leaf F_y is also $m - n$. If $\dim N = 1$, then we may assume that $N \subset \mathbb{R}$, and the leaves F_y are the level hypersurfaces of the function

$$y = f(x^1, \ldots, x^p)$$

defining the mapping $M \to \mathbb{R}$.

c) $m = n$. In this case, in a neighborhood of a regular point a, a mapping f is *bijective*. The tangent subspaces $T_a(M)$ and $T_b(N)$ to the manifolds M and N at the points a and b are of the same dimension, and the mapping f induces a nondegenerate linear map $f_* : T_a(M) \to T_b(N)$ with the matrix $\mathcal{M}_a$.

Note also that if $m < n$, then in a neighborhood of a regular point a the correspondence between the manifolds M and $f(M)$ is bijective.

1.2.3 Exterior Algebra, Pfaffian Forms, and the Cartan Lemma. Let x^i be coordinates in a neighborhood of a point a of the manifold M^n, and let $f(x)$ be a differentiable function defined in this neighborhood. Then the differential of this function can be written in the form

$$df = \frac{\partial f}{\partial x^i} dx^i. \tag{1.28}$$

The latter expression is a linear differential form in a coordinate neighborhood of the manifold M^n. However, this form is of special type because its coefficients are partial derivatives of the function $f(x)$. A linear differential form of general type can be written in the form

$$\theta = a_i dx^i. \tag{1.29}$$

Its coefficients $a_i = a_i(x)$ are coordinates of a differentiable covector field defined on the manifold M^n. The set of all linear forms on the manifold M^n is denoted by $\Lambda^1(M^n)$.

For the linear forms, the operations of addition and multiplication by a function can be defined in a natural way. In addition, for two linear forms θ_1 and θ_2, the operation of exterior multiplication $\theta_1 \wedge \theta_2$ can be defined. This operation is linear with respect to each factor and is anticommutative: $\theta_2 \wedge \theta_1 = -\theta_1 \wedge \theta_2$. The product $\theta_1 \wedge \theta_2$ is an *exterior quadratic form*. The exterior quadratic forms of general type are obtained by means of linear combinations of the exterior products of linear forms. The linear operations can be defined in a natural way in the set of exterior quadratic forms, and this set is a module over the ring of smooth functions on the manifold M^n. This module is denoted by $\Lambda^2(M^n)$ (see, for example, the book [KN 63] by Kobayashi and Nomizu, pp. 5–7). The localization of this module over each coordinate neighborhood $U \subset M^n$ is a free module with $\binom{n}{2} = \frac{n(n-1)}{2}$ generators. At each point, the exterior quadratic forms form a vector space Λ^2 of dimension $\frac{n(n-1)}{2}$ over the field of real or complex numbers.

In a similar manner, one can define the exterior differential forms of degree p, $p \leq n$ on the manifold M^n, and these forms generate a module $\Lambda^p(M^n)$ over the same ring. The localization of this module over each neighborhood $U \subset M^n$ is a free module of dimension $\binom{n}{p}$.

The multiplication of exterior forms of different degrees can be also defined. If θ_1 and θ_2 are exterior forms of degrees p and q, respectively, then their exterior product $\theta_1 \wedge \theta_2$ is an exterior form of degree $p + q$. This product satisfies the following property:

$$\theta_1 \wedge \theta_2 = (-1)^{pq} \theta_2 \wedge \theta_1. \tag{1.30}$$

By the skew symmetry, the exterior forms of degree greater than n vanish.

The exterior forms of different degrees form the *exterior algebra* on the manifold M^n:

$$\Lambda = \Lambda^0 + \Lambda^1 + \Lambda^2 + \ldots + \Lambda^n; \tag{1.31}$$

here Λ^p is the module of exterior forms of degree p. In particular, Λ^0 is the ring of differentiable functions on the manifold M^n. Exterior forms of degree p are also called *p-forms*, and 1-forms are also called the *Pfaffian forms.*

We now consider an exterior differential form of degree two on a manifold M^n. In terms of the coordinates x^i, this form can be written as

$$\theta = a_{ij} dx^i \wedge dx^j, \quad i, j = 1, \ldots, n,$$

where $a_{ij} = a_{ij}(x)$, $a_{ij} = -a_{ji}$, and $dx^i \wedge dx^j$ are the basis 2-forms. A skew-symmetric bilinear form is associated with the form θ. The bilinear form $\theta(\xi, \eta)$,

$$\theta(\xi, \eta) = a_{ij} \xi^i \eta^j,$$

determines the value of the form θ on a pair of vector fields ξ and η defined in $T(M^n)$. If these two vector fields satisfy the equation

$$\theta(\xi, \eta) = 0,$$

then we say that they are *in involution* with respect to the exterior quadratic form θ. The notion of the value of an exterior p-form on a system consisting of p vector fields given on the manifold M^n can be defined in a similar manner.

Note further the following proposition of algebraic nature, which is called the *Cartan lemma.*

Lemma 1.1 (É. Cartan). *Suppose the linearly independent* 1*-forms* ω^1, ω^2, $\ldots, \omega^p$ *and the* 1*-forms* $\theta_1, \theta_2, \ldots, \theta_p$ *are connected by the relation*

$$\theta_1 \wedge \omega^1 + \ldots + \theta_p \wedge \omega^p = 0. \tag{1.32}$$

Then the forms θ_a *are linearly expressed in terms of the forms* ω^a *as follows:*

$$\theta_a = l_{ab} \omega^b, \tag{1.33}$$

where

$$l_{ab} = l_{ba}. \tag{1.34}$$

Proof. Because the forms $\omega^a,\ a = 1, \dots, p$, are linearly independent in the covector space T^*, by adding the forms $\omega^\xi,\ \xi = p+1, \dots, n$, we complete $\omega^1, \dots, \omega^p$ to a basis for T^*. Then

$$\theta_a = l_{ab}\omega^b + l_{a\xi}\omega^\xi.$$

Substituting this into relation (1.32), we obtain

$$l_{ab}\omega^a \wedge \omega^b + \omega^a \wedge l_{a\xi}\omega^\xi = 0,$$

which implies $l_{a\xi} = 0$ and $l_{ab} = l_{ba}$. □

Cartan's lemma is of pure algebraic nature. But if the forms ω^a and θ^a are given on a differentiable manifold M, then Cartan's lemma is also valid, and the quantities l_{ab} are smooth functions on M.

In the algebra of differential forms, another operation—the *exterior differentiation*—can be defined. For functions, i.e., exterior forms of degree zero, this operation coincides with ordinary differentiation, and for exterior forms of type

$$\theta = a dx^{i_1} \wedge \ldots \wedge dx^{i_p}, \tag{1.35}$$

this operation is defined by means of the formula:

$$d\theta = da \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}. \tag{1.36}$$

It is possible to prove that this operation is invariant under the change of variables (see, for example, Cartan's book [C 45], p. 34).

The operation of exterior differentiation defines a linear mapping of the space $\Lambda^p(M^n)$ into the space $\Lambda^{p+1}(M^n)$:

$$d : \Lambda^p \to \Lambda^{p+1}. \tag{1.37}$$

Using formula (1.36), the formula for differentiation of a product of two exterior forms can be proved. Namely, if the forms θ_1 and θ_2 have degrees p and q, respectively, then

$$d(\theta_1 \wedge \theta_2) = d\theta_1 \wedge \theta_2 + (-1)^p \theta_1 \wedge d\theta_2. \tag{1.38}$$

In addition, the following formula holds:

$$d(d\theta) = 0. \tag{1.39}$$

This formula is called the *Poincaré lemma.* In particular, for a function f on M^n we have $d(df) = 0$. Conversely, if ω is a 1-form given in a simply connected domain of a manifold M^n and such that $d\omega = 0$, then $\omega = df$.

A p-form θ satisfying the condition $d\theta = 0$ is called *closed*, and a p-form θ satisfying the condition $\theta = d\sigma$, where σ is a $(p-1)$-form, is called *exact*. Poincaré's theorem states that *if M^n is a p-connected manifold* (*i.e., in M^n every p-dimensional contour is homotopic to zero*), *then in M^n any closed p-form is exact.* This theorem follows from the p-dimensional Stokes theorem.

Note also that the operation of exterior differentiation, defined by formula (1.36) by means of coordinates, does not depend on the choice of coordinates on the manifold M^n, i.e., this operation is invariant; it commutes with the operation of coordinate transformation on the manifold M^n.

1.2.4 The Structure Equations of the General Linear Group. As an example, we apply the operation of exterior differentiation to derive the structure equations of the general linear group $\mathbf{GL}(n)$. In Section 1.1.1, invariant forms for this group were determined for the frame bundle $\mathcal{R}(L^n)$ of a vector space L^n and were written in the form (1.4). Applying exterior differentiation to equations (1.4) and using equations (1.36), we obtain

$$d\omega = da^{-1} \wedge da. \tag{1.40}$$

From relation (1.4) we find that

$$da = a\omega, \tag{1.41}$$

and because $aa^{-1} = I$, we have

$$da^{-1} = -a^{-1}da \cdot a^{-1} = -\omega a^{-1}. \tag{1.42}$$

Substituting expressions (1.41) and (1.42) into equation (1.40), we arrive at the equation

$$d\omega = -\omega \wedge \omega. \tag{1.43}$$

In coordinate form, this equation is written as

$$d\omega_j^i = -\omega_k^i \wedge \omega_j^k,$$

or, more often, as

$$d\omega_j^i = \omega_j^k \wedge \omega_k^i. \tag{1.44}$$

Equations (1.43) and (1.44) are called the *structure equations* or the *Maurer–Cartan equations* of the general linear group $\mathbf{GL}(n)$.

1.2.5 The Frobenius Theorem. Suppose that a system of linearly independent 1-forms θ^a, $a = p+1, \ldots, n$, is given on a manifold M^n. At each point x of the manifold M^n, this system determines a linear subspace Δ_x of the space $T_x(M^n)$ via the equations

$$\theta^a(\xi) = 0. \tag{1.45}$$

The dimension of this subspace is equal to p. A set of such p-dimensional subspaces Δ_x^p given at every point x of the manifold M^n is called a *p-dimensional distribution* and is denoted by $\Delta^p(M^n)$.

An *integral manifold* of a system of Pfaffian equations

$$\theta^a = 0 \tag{1.46}$$

is a submanifold V^q of dimension q, $q \leq p$, whose tangent subspace $T_x V^q$ at any point x belongs to the element Δ_x^p of the distribution $\Delta^p(M^n)$, $T_x V^q \subset \Delta^p(M^n)$.

It is easy to prove that the system (1.46) always possesses one-dimensional integral manifolds. If the system has integral manifolds of maximal possible dimension p which form a foliation on the manifold M^n, then we say that the system is *completely integrable*. This means that through any point $x \in M^n$, there passes a unique p-dimensional integral manifold V^p of the system (1.46). A necessary and sufficient condition for a system (1.46) to be completely integrable is given by the Frobenius theorem (see Kobayashi and Nomizu [KN 63], vol. 2, p. 323).

Theorem 1.2 (Frobenius). *System* (1.46) *is completely integrable if and only if the exterior differentials of the forms θ^a vanish by means of the equations of this system.*

Analytically this can be written as follows:

$$d\theta^a = \theta^b \wedge \theta_b^a, \tag{1.47}$$

where θ_b^a are some new 1-forms.

Note that structure equations (1.44) of the general linear group $\mathbf{GL}(n)$, which we found earlier, are conditions of complete integrability for the system of equations (1.8) defining the infinitesimal displacement of a frame of the space L^n.

Note also that if a system of forms ω_j^i is given and it depends on $\rho \leq n^2$ parameters and satisfies structure equations (1.44), then by Frobenius' theorem, this system uniquely (up to a transformation of the general linear group $\mathbf{GL}(n)$) determines a ρ-parameter family of frames $\mathcal{R}^\rho$ in the space L^n.

1.2.6 The Cartan Test. If system (1.46) is not completely integrable, then it could still possess integral manifolds of dimension $q < p$. We say that the system of Pfaffian equations (1.46) is *in involution* if at least one two-dimensional integral manifold V^2 passes through each one-dimensional integral manifold V^1 of this system, at least one three-dimensional integral manifold V^3 passes through each of its two-dimensional integral manifolds V^2, etc., and finally, at least one integral manifold V^q of dimension q passes through each integral manifold V^{q-1} of dimension $q-1$.

Later we will often apply the *Cartan test* for the system of Pfaffian equations (1.46) to be in involution.

To formulate the Cartan test, first, we note that if V^q is an integral manifold of system (1.46), then on this manifold not only system (1.46) vanishes but also the system

$$d\theta^a = 0. \tag{1.48}$$

A q-dimensional subspace Δ_x^q tangent to the integral manifold V^q is characterized by the fact that each of its vectors satisfies every equation of system (1.46), and each pair of its vectors is in involution relative to the exterior quadratic forms $d\theta^a$, i.e., the pair satisfies system (1.48). These vectors are called the *one-dimensional integral elements* of system (1.46).

Let ξ_1 be a one-dimensional integral element of system (1.46). A two-dimensional integral element passing through element ξ_1 is determined by a vector ξ_2 that, in addition to the system of equations (1.45), together with ξ_1 satisfy the system

$$d\theta^a(\xi_1, \xi_2) = 0. \tag{1.49}$$

If the vector ξ_1 is held fixed, system (1.49) is a linear homogeneous system for finding ξ_2. Denote the rank of this system by r_1. Suppose that ξ_2 is a solution of system (1.49). The vectors ξ_1 and ξ_2 determine a two-dimensional integral element E_2 of system (1.46). To find a three-dimensional integral element of this system, we should consider the system

$$d\theta^a(\xi_1, \xi_3) = 0, \quad d\theta^a(\xi_2, \xi_3) = 0. \tag{1.50}$$

Each vector ξ_3 satisfying equations (1.50), together with the vectors ξ_1 and ξ_2, determines a three-dimensional integral element E_3. Denote the rank of system (1.50) by r_2. Similarly we can construct integral elements $E_4, \ldots, E_q$. They are connected by the relation

$$\xi_1 = E_1 \subset E_2 \subset E_3 \subset \ldots \subset E_q.$$

Denote by r_k the rank of the system of type (1.50) defining a vector ξ_{k+1}, which is in involution with the previously defined vectors $\xi_1, \ldots, \xi_k$, and let

$$s_1 = r_1, \quad s_2 = r_2 - r_1, \quad \ldots, s_{q-1} = r_{q-1} - r_{q-2}.$$

Let s_q be the dimension of the subspace defined by a system of type (1.50) for finding a vector ξ_q. The integers $s_1, s_2, \ldots, s_q$ are called the *characters* of system (1.46), and the integer

$$Q = s_1 + 2s_2 + \ldots + qs_q$$

is called its *Cartan number*. The characters of the Pfaffian system (1.46) are connected by the inequalities

$$s_1 \geq s_2 \geq \ldots \geq s_q. \tag{1.51}$$

The left-hand sides of equations (1.48) are exterior products of some linear forms from which q forms are linearly independent and are the basis forms of the integral manifold V^q. Let us denote these 1-forms by ω^a, $a = 1, \ldots, q$. In addition, equations (1.48) contain forms ω^u whose number is equal to $s_1 + s_2 + \ldots + s_q$. Applying the procedure outlined in the proof of the Cartan lemma, one can express the forms ω^u as linear combinations of the forms ω^a. The number of independent coefficients in these linear combinations is called the *arbitrariness of the general integral element* and is denoted by the letter S.

If the manifold M and the distribution $\Delta^p(M)$ defined on M by the system of equations (1.46) are real analytic, then the following theorem is valid:

Theorem 1.3 (É. Cartan's Test). *For a system of Pfaffian equations* (1.46) *to be in involution, it is necessary and sufficient that the condition $Q = S$ holds. Moreover, its q-dimensional integral manifold V^q depends on s_k functions of k variables, where s_k is the last nonvanishing character in sequence* (1.51).

Note also that if system (1.46) of Pfaffian equations is not in involution, this does not mean that the system has no solution. The further investigation of this system is connected with its successive differential prolongations. Moreover, it can be proved that after a finite number of prolongations one obtains either a system in involution—and in this case there exists a solution of system (1.46)—or arrives at a contradiction proving that the system has no solution.

The reader can find a more detailed exposition of the theory of systems of Pfaffian equations in involution in the books [BCGGG 91] by Bryant, Chern, Gardner, Goldsmith, and Griffiths; [C 45] by Cartan; [Fi 48] by Finikov; [Gr 83] by Griffiths; [GJ 87] by Griffiths and Jensen; and [AG 93] by Akivis and Goldberg. Examples of application of Cartan's test can be found in the rest of this book.

1.2.7 The Structure Equations of a Differentiable Manifold. Let us find the structure equations of a differentiable manifold M^n. As we have already noted, if a differentiable function $f(x)$ is given on the manifold M^n, then in local coordinates x^i, the differential of this function can be written in form (1.28). The operators $\frac{\partial}{\partial x^i}$ of differentiation with respect to the coordinates x^i form a basis of the tangent space $T_x(M^n)$, called the *natural basis*. We view the differentials dx^i as the coordinates of a tangent vector $d = \frac{\partial}{\partial x^i} dx^i$ with respect to this basis. If we replace the natural basis $\{\frac{\partial}{\partial x^i}\}$ by an arbitrary basis $\{e_i\}$ of the space $T_x(M^n)$:

$$e_i = x_i^j \frac{\partial}{\partial x^j}, \quad \frac{\partial}{\partial x^i} = \widetilde{x}_i^j e_j, \tag{1.52}$$

where (x_i^j) and $(\widetilde{x}_i^j)$ are mutually inverse matrices, then we can expand the vector d as

$$d = e_j \widetilde{x}_i^j dx^i = \omega^j e_j, \tag{1.53}$$

where we used the notation

$$\omega^j = \widetilde{x}_i^j dx^i, \quad i, j = 1, \ldots, n. \tag{1.54}$$

The forms ω^j are called the *basis forms* of the manifold M^n.

Taking exterior derivatives of equations (1.54), we obtain

$$d\omega^i = d\widetilde{x}_j^i \wedge dx^j. \tag{1.55}$$

Eliminating the differentials dx^j by means of relations (1.54) from equations (1.55), we arrive at the equations:

$$d\omega^i = d\widetilde{x}_k^i \wedge x_j^k \omega^j. \tag{1.56}$$

Equations (1.56) imply that

$$d\omega^i = \omega^j \wedge \omega_j^i, \tag{1.57}$$

where the forms ω_j^i are not uniquely defined. In fact, subtracting (1.56) from (1.57), we find that

$$\omega^j \wedge (\omega_j^i + x_j^k d\widetilde{x}_k^i) = 0.$$

Applying the Cartan lemma to these equations, we obtain the equations

$$\omega_j^i + x_j^k d\widetilde{x}_k^i = x_{jk}^i \omega^k$$

or

$$\omega_j^i = -x_j^k d\widetilde{x}_k^i + x_{jk}^i \omega^k, \tag{1.58}$$

where $x_{jk}^i = x_{kj}^i$.

Equations (1.57) are the first set of structure equations of the manifold M^n. By the Frobenius theorem, it follows from equations (1.57) that the system of equations $\omega^i = 0$ is completely integrable. The first integrals of this system are the coordinates x^i of a point x of the manifold M^n.

Let us find the second set of the structure equations of the manifold M^n, which are satisfied by the forms ω^i_j. Exterior differentiation of equations (1.58) leads to the equations

$$d\omega^i_j = -dx^k_j \wedge d\widetilde{x}^i_k + dx^i_{jk} \wedge \omega^k + x^i_{jk}\omega^l \wedge \omega^k_l. \tag{1.59}$$

The entries of the matrices (x^j_i) and $(\widetilde{x}^j_i)$ are connected by the relation

$$x^k_j \widetilde{x}^i_k = \delta^i_j.$$

If we differentiate this relation, we find that

$$dx^k_j = -x^k_q x^l_j d\widetilde{x}^q_l.$$

Substituting these expressions for dx^k_j into equations (1.59) and using relations (1.58), we find that

$$d\omega^i_j = \omega^k_j \wedge \omega^i_k + (\nabla x^i_{jk} + x^p_{jl} x^i_{pk}\omega^l) \wedge \omega^k, \tag{1.60}$$

where ∇x^i_{jk} are defined according to the rule (1.15). Define also the 1-forms

$$\omega^i_{jk} = \nabla x^i_{jk} + x^p_{jl} x^i_{pk}\omega^l + x^i_{jkl}\omega^l, \tag{1.61}$$

where $x^i_{jkl} = x^i_{jlk}$. Using these equations, we can write equations (1.60) as

$$d\omega^i_j = \omega^k_j \wedge \omega^i_k + \omega^i_{jk} \wedge \omega^k. \tag{1.62}$$

These equations form the second set of structure equations of the manifold M^n.

Using the same procedure we just used to define the forms ω^i, ω^i_j, and ω^i_{jk} on the differentiable manifold M^n and to find structure equations for these forms, we can define higher-order forms $\omega^i_{jkl}, \ldots$ and find structure equations for them (see Laptev [Lap 66]). However, in this book we will not need these higher-order forms and equations.

As we already noted, the forms ω^i are basis forms of the manifold M^n. The forms ω^i_j are the fiber forms of the bundle $\mathcal{R}^1(M^n)$ of frames of first order over M^n, and the forms ω^i_{jk}, together with the forms ω^i_j, are the fiber forms of the bundle $\mathcal{R}^2(M^n)$ of frames of second order over M^n. The fibers $\mathcal{R}^1_x$ and $\mathcal{R}^2_x$ of these two fibrations are defined on the manifolds $\mathcal{R}^1(M^n)$ and $\mathcal{R}^2(M^n)$ by the equations $\omega^i = 0$.

We denote by δ the restriction of the differential d to the fibers $\mathcal{R}^1_x$ and $\mathcal{R}^2_x$ of the frame bundles under consideration. Let us also denote the restrictions of the forms ω^i_j and ω^i_{jk} to these bundles by $\pi^i_j = \omega^i_j(\delta)$ and $\pi^i_{jk} = \omega^i_{jk}(\delta)$ respectively. Then it follows from equations (1.62) that

$$\delta\pi^i_j = \pi^k_j \wedge \pi^i_k. \tag{1.63}$$

Equations (1.63) coincide with the structure equations (1.44) of the general linear group $\mathbf{GL}(n)$. Thus, the forms π^i_j are invariant forms of the group $\mathbf{GL}(n)$ of admissible transformations of the first-order frames $\{e_i\}$ associated with a point x of the manifold M^n, and the fiber $\mathcal{R}^1_x$ is diffeomorphic to this group. This fiber is an orbit of a vector of a representation space of the group $\mathbf{GL}(n)$.

This and relations (1.5) show that if $\omega^i = 0$, the vectors e_i composing a frame in the space $T_x(M^n)$ satisfy the equations

$$\delta e_i = \pi^j_i e_j,$$

and the forms ω^i composing a coframe satisfy the equations

$$\delta\omega^i = -\pi^i_j \omega^j. \tag{1.64}$$

Next, consider the forms $\pi^i_{jk} = \omega^i_{jk}(\delta)$. Relations (1.61) imply that

$$\pi^i_{jk} = \nabla_\delta x^i_{jk},$$

and thus $\pi^i_{jk} = \pi^i_{kj}$. It is not so difficult to show that the forms π^i_{jk} satisfy the following structure equations

$$\delta\pi^i_{jk} = \pi^l_{jk} \wedge \pi^i_l + \pi^l_j \wedge \pi^i_{lk} + \pi^l_k \wedge \pi^i_{jl}$$

(see Laptev [Lap 66]) and that these forms together with the forms π^i_j are invariant forms of the group $\mathbf{GL}^2(n)$ of admissible transformations of the second-order frames associated with the point $x \in M^n$. The group $\mathbf{GL}^2(n)$ is diffeomorphic to the fiber $\mathcal{R}^2_x$.

1.2.8 Affine Connections on a Differentiable Manifold. In what follows we will use the notion of an affine connection in a frame bundle. An *affine connection* γ on a manifold M^n is defined in the frame bundle $\mathcal{R}^2(M^n)$ by means of an invariant horizontal distribution Δ defined by a system of Pfaffian forms

$$\theta^i_j = \omega^i_j - \Gamma^i_{jk}\omega^k \tag{1.65}$$

vanishing on Δ. The distribution Δ is invariant with respect to the group of affine transformations acting in $\mathcal{R}^1(M^n)$.

Using equations (1.65), we eliminate the forms ω_j^i from equations (1.57). As a result, we obtain

$$d\omega^i = \omega^j \wedge \theta_j^i + R_{jk}^i \omega^j \wedge \omega^k, \tag{1.66}$$

where $R_{jk}^i = \Gamma_{[jk]}^i$. The condition for the distribution Δ to be invariant leads to the following equations:

$$d\theta_j^i = \theta_j^k \wedge \theta_k^i + R_{jkl}^i \omega^k \wedge \omega^l. \tag{1.67}$$

The Pfaffian forms $\theta = (\theta_j^i)$ with their values in the Lie algebra $\mathrm{gl}(n)$ of the group $\mathbf{GL}(n)$ are called the *connection forms* of the connection γ.

The quantities R_{jk}^i and R_{jkl}^i form tensors called the *torsion tensor* and the *curvature tensor* of the connection γ, respectively.

Conversely, one can prove that if in the frame bundle $\mathcal{R}^2(M^n)$, the forms θ_j^i are given, and these forms together with the forms ω^i satisfy equations (1.66) and (1.67), then the forms θ_j^i define an affine connection γ on M^n, and the tensors R_{jk}^i and R_{jkl}^i are the torsion and curvature tensors of this connection γ.

As a rule, in our considerations the torsion-free affine connections will arise for which $R_{jk}^i = 0$. For these connections, the form $\omega = (\omega_j^i)$ can be chosen as a connection form. Under this assumption, the structure equations (1.66) and (1.67) can be written in the form

$$d\omega^i = \omega^j \wedge \omega_j^i, \;\; d\omega_j^i = \omega_j^k \wedge \omega_k^i + R_{jkl}^i \omega^k \wedge \omega^l. \tag{1.68}$$

A more detailed presentation of the foundations of the theory of affine connections can be found in the books [KN 63] by Kobayashi and Nomizu and [Lich 55] by Lichnerowicz (see also the papers [Lap 66, 69] by Laptev).

1.3 Projective Space

1.3.1 Projective Transformations, Projective Frames, and the Structure Equations of a Projective Space. We assume that the reader is familiar with the notions of the projective plane and the three-dimensional projective space. These notions can be generalized for the multidimensional case in a natural way (see Dieudonné [Di 64]).

Consider an $(n+1)$-dimensional vector space L^{n+1}. Denote by $\widetilde{L}^{n+1}$ the set of all nonzero vectors of the space L^{n+1}. We consider collinear vectors of $\widetilde{L}^{n+1}$ to be equivalent and define the *n-dimensional projective space* $\mathbb{P}^n$ as the quotient of the set $\widetilde{L}^{n+1}$ by this equivalence relation: $\mathbb{P}^n = \widetilde{L}^{n+1}/\{0\}$.

This means that a point of $\mathbb{P}^n$ is a collection of nonzero collinear vectors λx of L^{n+1}, i.e., a point of $\mathbb{P}^n$ is a one-dimensional subspace of L^{n+1}. A straight line of $\mathbb{P}^n$ is a two-dimensional subspace of L^{n+1}, etc. If in L^{n+1} a basis defined by the vectors $e_0, e_1, \ldots, e_n$ is given, then any vector $x \neq 0$ of L^{n+1} can be decomposed relative to this basis:

$$x = x^0 e_0 + x^1 e_1 + \ldots + x^n e_n,$$

where the numbers $x^0, x^1, \ldots, x^n$ are the coordinates of the vector x relative to the basis $\{e_i\}$. In the space L^{n+1}, a set of collinear vectors corresponds to the point x of $\mathbb{P}^n$, and the coordinates of this set are the numbers $(\lambda x^0, \lambda x^1, \ldots, \lambda x^n)$, where $\lambda \neq 0$. These numbers are called the *homogeneous coordinates* of the point $x \in \mathbb{P}^n$. Note that they are unique up to a multiplicative factor.

Linear transformations of the space L^{n+1} give rise to corresponding *projective transformations* of the space $\mathbb{P}^n$. Under these transformations, straight lines are transformed into straight lines, planes into planes, etc. Because a point in $\mathbb{P}^n$ is defined by homogeneous coordinates, transformations of the form

$$y^u = \rho x^u, \quad \rho \neq 0, \quad u = 0, 1, \ldots, n,$$

define the *identity transformation* of the space $\mathbb{P}^n$. Thus, the projective transformations can be written as

$$\rho y^u = a^u_v x^v, \quad \rho \neq 0, \quad u, v = 0, 1, \ldots, n,$$

where $\det(a^u_v) \neq 0$. Therefore, the *group of projective transformations* of the space $\mathbb{P}^n$ depends on $(n+1)^2 - 1 = n^2 + 2n$ parameters. This group is denoted by $\mathbf{PGL}(n)$.

A *projective frame* in the space $\mathbb{P}^n$ is a system consisting of $n+1$ points A_u, $u = 0, 1, \ldots, n$, and a *unity point* E, which are in general position. In the space L^{n+1}, to the points A_u there correspond linearly independent vectors e_u, and the vector $e = \sum_{u=0}^n e_u$ corresponds to the point E. These vectors are defined in L^{n+1} up to a common factor. It follows that the set of projective frames $\{A_u\}$ depends on $n^2 + 2n$ parameters. We shall assume that the unity point E is given along with the basis points A_u, although we might not mention it on every occasion.

We will perform the linear operations on points of a projective space $\mathbb{P}^n$ via the corresponding vectors in the space L^{n+1}. These operations will be invariant in $\mathbb{P}^n$ if we multiply all corresponding vectors in L^{n+1} by a common factor.

In some instances, we assume that a vectorial frame in L^{n+1} is normalized by the condition

$$e_0 \wedge e_1 \wedge \ldots \wedge e_n = \begin{cases} \pm 1 & \text{for odd } n, \\ 1 & \text{for even } n, \end{cases} \tag{1.69}$$

where the wedge denotes the exterior product. Condition (1.69) can always be achieved by multiplying all vectors of the frame by an appropriate factor.[1] Hence the group $\mathbf{PGL}(n)$ is isomorphic to the quotient group $\mathbf{SL}(n+1)/C_2$, where $C_2 = \{1, -1\}$ is the cyclic group of the second degree.

When such a normalization has been done, the vectors of a frame in L^{n+1} corresponding to the point of a projective frame $\{A_u\}$ are uniquely determined. Thus, the group of projective transformations of the space $\mathbb{P}^n$ is isomorphic to the special linear group $\mathbf{SL}(n+1)$ of transformations of L^{n+1}. Sometimes we will write the normalization condition (1.69) in the form

$$A_0 \wedge A_1 \wedge \ldots \wedge A_n = \pm 1. \tag{1.70}$$

The equations of infinitesimal displacement of a frame in $\mathbb{P}^n$ have the same form (1.5) as in L^n:

$$dA_u = \omega_u^v A_v, \tag{1.71}$$

but now the indices u and v take the values from 0 to n, and by condition (1.70), the forms ω_u^v in equations (1.71) are connected by the relation

$$\omega_0^0 + \omega_1^1 + \ldots + \omega_n^n = 0. \tag{1.72}$$

This condition shows that the number of linearly independent forms ω_u^v becomes equal to the number of parameters on which the group $\mathbf{PGL}(n)$ of projective transformations of the space $\mathbb{P}^n$ depends.

The structure equations of the space $\mathbb{P}^n$ have the same form as they had in the space L^n:

$$d\omega_v^u = \omega_v^w \wedge \omega_w^u, \tag{1.73}$$

but now we have a new range for the indices u, v and w: $u, v, w = 0, 1 \ldots, n$.

It is well known that a projective space $\mathbb{P}^n$ is a differentiable manifold. Let us show that equations (1.73) are a particular case of the structure equations (1.57) and (1.62) of a differentiable manifold. To show this, first we write equations (1.73) for $v = 0$ and $u = i$, where $i = 1, \ldots, n$, in the form

$$d\omega_0^i = \omega_0^j \wedge \theta_j^i,$$

[1] If we multiply all the vectors $e_0, e_1, \ldots, e_n$ by λ, then the determinant is multiplied by λ^{n+1}. Thus if n is odd, then it is impossible to change the determinant sign.

where $\theta_j^i = \omega_j^i - \delta_j^i \omega_0^0$. These equations differ from equations (1.57) only in notation. Next, taking exterior derivatives of the forms θ_j^i and applying equations (1.73), we find that

$$d\theta_j^i = \theta_j^k \wedge \theta_k^i + (\delta_k^i \omega_j^0 + \delta_j^i \omega_k^0) \wedge \omega_0^k.$$

Comparing these equations with equations (1.62), we observe that they coincide if

$$\omega_{jk}^i = \delta_k^i \omega_j^0 + \delta_j^i \omega_k^0.$$

The latter relations prove that if a first-order frame is held fixed, the second-order frames of a projective space $\mathbb{P}^n$ depend on n parameters while on a general differentiable manifold they depend on n^3 parameters.

Note that the forms ω_0^i constitute a basis in the cotangent space $T_x^*(\mathbb{P}^n)$ of a projective space $\mathbb{P}^n$. The corresponding basis in the tangent space $T_x(\mathbb{P}^n)$ is formed by the vectors $\boldsymbol{v}_i$ which are directed along the lines A_0A_i (see Griffiths and Harris [GH 79]). In what follows, we will denote the forms ω_0^i by ω^i.

1.3.2 The Duality Principle. Consider a hyperplane ξ in $\mathbb{P}^n$. The equations of this hyperplane can be written in the form

$$\xi_u x^u = 0, \quad u = 0, 1, \ldots, n, \tag{1.74}$$

where the coefficients ξ_u are defined up to a constant factor. These coefficients can be viewed as homogeneous coordinates of the hyperplane ξ. They are called the *tangential coordinates* of the hyperplane ξ. This consideration shows that the collection of hyperplanes of a projective space $\mathbb{P}^n$ is a new projective space of the same dimension n. This space is denoted by $(\mathbb{P}^n)^*$ and called *dual* to the space $\mathbb{P}^n$.

Equation (1.74) is the condition of the incidence of a point x with coordinates x^u and a hyperplane $\xi = (\xi_u)$. Denote the left-hand side of equation (1.74) by (ξ, x). Then we can write it in the form

$$(\xi, x) = 0. \tag{1.75}$$

Equation (1.75) shows that the spaces $\mathbb{P}^n$ and $(\mathbb{P}^n)^*$ are mutually dual, that is, the space $\mathbb{P}^n$ is dual to the space $(\mathbb{P}^n)^*$,

$$((\mathbb{P}^n)^*)^* = \mathbb{P}^n.$$

The passage from the space $\mathbb{P}^n$ to the space $(\mathbb{P}^n)^*$ (or back from $(\mathbb{P}^n)^*$ to $\mathbb{P}^n$) is called the *duality principle*. Let $\mathbb{P}^m \subset \mathbb{P}^n$ be an m-dimensional subspace of the space $\mathbb{P}^n$. Then $\mathbb{P}^m$ is spanned by $m+1$ linearly independent points $M_0, M_1, \ldots, M_m$. By the duality principle, to every point M_i,

$i = 0, 1, \dots, m$, there corresponds a hyperplane μ^i. Hence to a subspace $\mathbb{P}^m$ there corresponds in $(\mathbb{P}^n)^*$ a subspace of dimension $n - m - 1$, which is the intersection of hyperplanes μ^i. Therefore, $(\mathbb{P}^m)^* = \mathbb{P}^{n-m-1} = \mu^0 \wedge \mu^1 \wedge \dots \wedge \mu^m$.

If in the space $\mathbb{P}^n$ we have $\mathbb{P}_1 \subset \mathbb{P}_2 \subset \mathbb{P}^n$, then in $(\mathbb{P}^n)^*$ we have $P_2^* \subset P_1^* \subset (\mathbb{P}^n)^*$. This means that the duality principle reverses the incidence of subspaces in the spaces $\mathbb{P}^n$ and $(\mathbb{P}^n)$. Thus, to each theorem of projective geometry and to any configuration composed from subspaces of the space $\mathbb{P}^n$, there corresponds a dual theorem and configuration in the space $(\mathbb{P}^n)^*$ (see Rosenfeld [Ro 97], p. 135).

The mapping $\mathcal{C}$ of the space $\mathbb{P}^n$ to the space $(\mathbb{P}^n)^*$ preserving the incidence of subspaces is called the *correlation*, $\mathcal{C} : \mathbb{P}^n \to (\mathbb{P}^n)^*$, where $\xi = \mathcal{C}x$ is a nondegenerate linear mapping. In a frame $\{A_u\}$, $u = 0, \dots, n$, of the space $\mathbb{P}^n$, the correlation $\mathcal{C}$ can be written in the form

$$\xi_u = c_{uv} x^v, \quad \det(c_{uv}) \neq 0, \tag{1.76}$$

where x^v are point coordinates and ξ_u are tangential coordinates in $\mathbb{P}^n$.

In the space $(\mathbb{P}^n)^*$, let us choose a coframe consisting of $n+1$ hyperplanes α^u connected with the points of the frame $\{A_u\}$ by the following condition:

$$(\alpha^u, A_v) = \delta_v^u. \tag{1.77}$$

This coframe is called *dual* to the frame $\{A_u\}$. Condition (1.77) means that the hyperplane α^u contains all points A_v, $v \neq u$, and that the condition of normalization $(\alpha^u, A_u) = 1$ holds.

We write the equations of infinitesimal displacement of the tangential frame $\{\alpha^u\}$ in the form

$$d\alpha^u = \widetilde{\omega}_v^u \alpha^v, \quad u, v = 0, 1, \dots, n. \tag{1.78}$$

Differentiating relations (1.77) and using equations (1.71), (1.78), and (1.77), we arrive at the equations

$$\omega_v^u + \widetilde{\omega}_v^u = 0,$$

from which it follows that equations (1.78) take the form

$$d\alpha^u = -\omega_v^u \alpha^v. \tag{1.79}$$

The structure equations (1.73) are the conditions for complete integrability of both equations (1.71) of infinitesimal displacement of a point frame and equations (1.79) of infinitesimal displacement of a tangential frame. Thus, if the 1-forms ω_v^u depend on some number ρ, $\rho \leq n^2 + n$, of parameters, and satisfy structure equations (1.73), then in the space $\mathbb{P}^n$, they define a ρ-parameter family of frames, up to a projective transformation of $\mathbb{P}^n$. The location of this family of frames is completely determined by the location of a frame corresponding to initial values of parameters. Conversely, if in $\mathbb{P}^n$ a family of projective frames that depends on ρ parameters is given, then the components ω_v^u of infinitesimal displacement of this family are unchanged under its projective transformation. Similarly, the 1-forms ω_v^u define a ρ-parameter family of coframes $\{\alpha_u\}$, up to a projective transformation. Hence, the forms ω_v^u are invariant forms with respect to transformations of the projective group.

1.3.3 Projectivization. In what follows, we will often use a special construction called the *projectivization.*

Let $\mathbb{P}^n$ be a projective space of dimension n, and let $\mathbb{P}^m$ be a subspace of dimension m, where $0 \leq m < n$. We say that two points $x, y \in \mathbb{P}^n$, $x, y \notin \mathbb{P}^m$, are *in the relation* $\mathbb{P}^m$ and write this as $x\mathbb{P}^m y$ if the straight line xy intersects the subspace $\mathbb{P}^m$. It is easy to check that the introduced relation is an equivalence relation. Thus, the points in the relation $\mathbb{P}^m$ are called $\mathbb{P}^m$-*equivalent.* This equivalence relation divides all points of the space $\mathbb{P}^n$ into the equivalence classes in such a way that all points of an $(m+1)$-plane $\mathbb{P}^{m+1}$ containing the subspace $\mathbb{P}^m$ belong to one class.

The equivalence relation introduced above allows us to factorize the space $\mathbb{P}^n$ by this relation. The resulting quotient space $\mathbb{P}^n/\mathbb{P}^m$ is called the *projectivization* of $\mathbb{P}^n$ with the *center* $\mathbb{P}^m$ and denoted by $\widetilde{\mathbb{P}}^{n-m-1}$:

$$\widetilde{\mathbb{P}}^{n-m-1} = \mathbb{P}^n/\mathbb{P}^m.$$

Sometimes the quotient space $\mathbb{P}^n/\mathbb{P}^m$ is called the *factorization* of $\mathbb{P}^n$ with respect to $\mathbb{P}^m$. The projectivization $\widetilde{\mathbb{P}}^{n-m-1}$ is a projective space of dimension $n-m-1$. Let us take a basis in $\mathbb{P}^n$ in such a way that its points A_i, $i = 0, 1, \ldots, m$, belong to the center $\mathbb{P}^m$ of projectivization. Then the basis of the space $\widetilde{\mathbb{P}}^{n-m-1}$ is formed by the points $\widetilde{A}_\alpha = A_\alpha/P^m$, $\alpha = m+1, \ldots, n$. Because the center $\mathbb{P}^m$ is unchanged under projectivization, the equations of infinitesimal displacement of the frame $\{A_i, A_\alpha\}$ of the space $\mathbb{P}^n$ can be written in the form

$$dA_i = \omega_i^j A_j, \;\; dA_\alpha = \omega_\alpha^\beta A_\beta + \omega_\alpha^i A_i.$$

Thus, in this family of frames we have $\omega_i^\alpha = 0$. Hence, the structure equations (1.73) of a projective space $\mathbb{P}^n$ imply that

$$d\omega_\alpha^\beta = \omega_\alpha^\gamma \wedge \omega_\gamma^\beta. \tag{1.80}$$

This allows us to consider the forms ω_α^β as the components of infinitesimal displacement of the frame $\{\widetilde{A}_\alpha\}$ of the projectivization $\widetilde{\mathbb{P}}^{n-m-1}$, so that

$$d\widetilde{A}_\alpha = \omega_\alpha^\beta \widetilde{A}_\beta.$$

On some occasions, we will identify the points $\widetilde{A}_\alpha$ of the projectivization $\widetilde{\mathbb{P}}^{n-m-1}$ with the points A_α of the projective space $\mathbb{P}^n$.

Note that one can also consider the projectivization of a vector space L^m by its 0-dimensional subspace $\{0\}$. The result of this projectivization is the subspace $\mathbb{P}^{m-1} = L^m/\{0\}$. Actually, in the definition of the projective space $\mathbb{P}^n$ itself (see Section 1.3.1), we already used the projectivization, so that $\mathbb{P}^n = L^{n+1}/\{0\}$.

1.3.4 Classical Homogeneous Spaces (Affine, Euclidean, Non-Euclidean) and Their Transformations. As was noted in the Preface, a projective space can be used to represent all classical homogeneous spaces: affine, Euclidean, non-Euclidean, conformal, and other spaces. To do this, one fixes certain invariant objects in a projective space $\mathbb{P}^n$ and reduces the group of transformations of the space by requiring that they be invariant. Now we show how this can be carried out for the basic homogeneous spaces.

An *affine space* $\mathbb{A}^n$ is a projective space $\mathbb{P}^n$ in which a hyperplane α is fixed. This hyperplane is called the *ideal hyperplane* or the *hyperplane at infinity* (or the *improper hyperplane*). The *affine transformations* are those projective transformations that transform this hyperplane into itself. Straight lines of the space $\mathbb{P}^n$ that intersect the ideal hyperplane at the same point are called *parallel straight lines* of the space $\mathbb{A}^n$. Two-dimensional planes of $\mathbb{P}^n$ intersecting the ideal hyperplane along the same straight line are called *parallel 2-planes* of the space $\mathbb{A}^n$, etc.

As a frame in the space $\mathbb{A}^n$, it is natural to take a projective frame whose points $A_1, \ldots, A_n$ lie in the ideal hyperplane α. The equations of infinitesimal displacement of such a frame have the form

$$\begin{cases} dA_0 = & \omega_0^0 A_0 + \quad \omega_0^i A_i, \\ dA_i = & \qquad\qquad \omega_i^j A_j, \; i, j = 1, \ldots, n. \end{cases} \tag{1.81}$$

Equations (1.81) show that in this case the forms ω_i^0 in equations (1.71) are equal to zero: $\omega_i^0 = 0$. This and structure equations (1.73) imply that $d\omega_0^0 = 0$. Thus, the form ω_0^0 is a total differential: $\omega_0^0 = d\log|\lambda|$. Substituting this value of the form ω_0^0 into the first equation of (1.81), we find that

$$dA_0 = \frac{d\lambda}{\lambda} A_0 + \omega_0^i A_i.$$

It follows that

$$d\Big(\frac{A_0}{\lambda}\Big) = \omega_0^i \, \frac{A_i}{\lambda}. \tag{1.82}$$

If we set

$$\frac{A_0}{\lambda} = x, \quad \frac{A_i}{\lambda} = e_i, \tag{1.83}$$

then equation (1.82) can be written as

$$dx = \omega_0^i e_i. \tag{1.84}$$

Differentiating the second equation of (1.83), we obtain

$$de_i = \theta_i^j e_j, \tag{1.85}$$

where

$$\theta_i^j = \omega_i^j - \delta_i^j \, d\ln|\lambda|.$$

We may consider the point x as the vertex of an affine frame and the vectors e_i as its basis vectors. Equations (1.84) and (1.85) are the equations of infinitesimal displacement of this affine frame $\{x, e_i\}$. These equations contain $n + n^2$ linearly independent forms ω_0^i and θ_i^j. This corresponds to the fact that the group of affine transformations of the space $\mathbb{A}^n$ depends on $n+n^2$ parameters. The forms ω_0^i determine a parallel displacement of the frame, and the forms θ_i^j determine the isotropy transformations of this frame, which keep the point x invariant.

The structure equations of the space $\mathbb{A}^n$ can be obtained from equations (1.73). In fact, we derive from those equations that

$$\begin{aligned} d\omega_0^i &= \omega_0^0 \wedge \omega_0^i + \omega_0^j \wedge \omega_j^i = \omega_0^j \wedge \theta_j^i, \\ d\theta_j^i &= d\omega_j^i = \omega_j^k \wedge \omega_k^i = \theta_j^k \wedge \theta_k^i. \end{aligned}$$

As a result, the structure equations of the affine space $\mathbb{A}^n$ have the form

$$\left\{ \begin{aligned} d\omega_0^i &= \omega_0^j \wedge \theta_j^i, \\ d\theta_j^i &= \theta_j^k \wedge \theta_k^i. \end{aligned} \right. \tag{1.86}$$

These equations imply that the isotropy transformations form an invariant subgroup in the group of affine transformations of the space $\mathbb{A}^n$, and this subgroup is isomorphic to the general linear group $\mathbf{GL}(n)$. In addition, equations (1.86) imply that the parallel displacements form a subgroup that is not an invariant subgroup.

A *Euclidean space* $\mathbb{E}^n$ is obtained from an affine space $\mathbb{A}^n$ if in the ideal hyperplane of the latter space a nondegenerate imaginary quadric Q of dimension $n-2$ is fixed. The equations of this quadric Q can be written in the form

$$x^0 = 0, \;\; \sum_{i=1}^n (x^i)^2 = 0. \tag{1.87}$$

The *Euclidean transformations* are those affine transformations that transform this quadric into itself.

The quadric Q allows us to define the scalar product (a, b) of vectors a and b in the Euclidean space $\mathbb{E}^n$. If we take vectors of a frame in such a way that

$$(e_i, e_j) = \delta_{ij} \tag{1.88}$$

(here δ_{ij} is the Kronecker symbol: $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$), then the forms θ^i_j from equations (1.85) are connected by the relations

$$\theta^j_i + \theta^i_j = 0, \tag{1.89}$$

which are obtained by differentiating equations (1.88). The number of independent forms in equations (1.84) and (1.85) is now equal to $n + \frac{1}{2}n(n-1) = \frac{1}{2}n(n+1)$. This number coincides with the number of parameters on which the group of motions of space $\mathbb{E}^n$ depends. The structure equations of the space $\mathbb{E}^n$ still have the form (1.86).

A *non-Euclidean space* is a projective space $\mathbb{P}^n$ in which a nondegenerate invariant hyperquadric

$$Q(X, X) = g_{uv}x^u x^v = 0, \quad u, v = 0, 1, \ldots, n, \tag{1.90}$$

is fixed. Suppose for definiteness that a non-Euclidean space is elliptic, i.e. the hyperquadric $Q(X, X)$ is positive definite. Then we may choose the points of a projective frame $\{A_u\}$ in such a way that they form an autopolar simplex with respect to this hyperquadric, and we normalize the vertices of this simplex. This means that we have

$$Q(A_u, A_v) = \delta_{uv}, \tag{1.91}$$

and the forms ω^v_u from equations (1.71) satisfy the equations

$$\omega^v_u + \omega^u_v = 0. \tag{1.92}$$

The *elliptic transformations* are those projective transformations of the space $\mathbb{P}^n$ that preserve the hyperquadric Q. These transformations depend on $\frac{1}{2}n(n+1)$ parameters, and the latter number coincides with the number of independent forms among the forms ω^u_v.

If the hyperquadric Q is of signature $(1, n)$, then it defines the *hyperbolic geometry* in $\mathbb{P}^n$, which is also called the *Lobachevsky geometry.*

1.4 Specializations of Moving Frames

1.4.1 The First Specialization. In our study of the structure of submanifolds in a projective space, we will often apply the method of specialization of moving frames. The idea of this method is that from all projective frames associated with an element of a submanifold, we will take the frames that are most closely connected with the element and its differential neighborhood of a certain order. Such a specialization can be conducted analytically and geometrically.

Consider, for example, how the method of specialization of moving frames applies in the study of geometry of a curve on a projective plane. In this study, we will use both geometric and analytic variations of this method.

Let Γ be a smooth simple connected curve in the projective plane $\mathbb{P}^2$. A moving frame in $\mathbb{P}^2$ consists of three points A_0, A_1, and A_2 that do not belong to a straight line. The equations of infinitesimal displacements of such a frame have the form

$$dA_u = \omega_u^v A_v, \quad u, v = 0, 1, 2, \tag{1.93}$$

where ω_u^v are differential 1-forms satisfying the structure equations of the plane $\mathbb{P}^2$:

$$d\omega_u^v = \omega_u^w \wedge \omega_w^v, \quad u, v, w = 0, 1, 2 \tag{1.94}$$

(cf. equations (1.71) and (1.73)).

We assume that the family of projective frames in $\mathbb{P}^2$ is normalized by the condition

$$A_0 \wedge A_1 \wedge A_2 = 1 \tag{1.95}$$

(cf. equation (1.70)). Differentiating (1.95) with the help of (1.93) and using the fact that the points A_0, A_1, and A_2 are linearly independent, we obtain that

$$\omega_0^0 + \omega_1^1 + \omega_2^2 = 0 \tag{1.96}$$

(cf. equation (1.72)).

First, we apply two geometric specializations of the moving frame. Suppose that $A_0 = x \in \Gamma$ and locate the point A_1 on the tangent $T_x(\Gamma)$ to Γ at the point x, $A_1 \in T_x(\Gamma)$. Then we have

$$dA_0 = \omega_0^0 A_0 + \omega_0^1 A_1. \tag{1.97}$$

Comparing (1.97) with the first equation of (1.93), we see that

$$\omega_0^2 = 0. \tag{1.98}$$

The form ω_0^1 is a basis form on the curve Γ: if $\omega_0^1 = 0$, then the point A_0 is a fixed point on Γ. This form ω_0^1 is proportional to the differential du of a parameter u moving the point $x = A_0$ along Γ: $\omega_0^1 = \alpha du$. The parameter u is called the *principal parameter.*

Taking the exterior derivative of equation (1.98) with the help of (1.94), we obtain the exterior quadratic equation

$$\omega_0^1 \wedge \omega_1^2 = 0,$$

which by Cartan's lemma implies that

$$\omega_1^2 = b_2 \omega_0^1. \tag{1.99}$$

Note that the coefficient b_2 is defined in a second-order differential neighborhood of the point x. In what follows, the subindex will denote an order of a neighborhood in which an object in question is defined.

If $b_2 = 0$ at all points of Γ, then

$$dA_0 = \omega_0^0 A_0 + \omega_0^1 A_1, \;\; dA_1 = \omega_1^0 A_0 + \omega_1^1 A_1,$$

and

$$d(A_0 \wedge A_1) = (\omega_0^0 + \omega_1^1)(A_0 \wedge A_1),$$

and the curve Γ becomes the straight line $A_0 \wedge A_1$. In what follows, we will assume that

$$b_2 \neq 0, \tag{1.100}$$

i.e., that *the curve* Γ *is not a straight line.*

For the next specialization of the moving frame, we apply the analytic method. Taking the exterior derivative of (1.99) with the help of (1.94) and (1.96), we obtain the exterior quadratic equation

$$(db_2 - 3b_2\omega_1^1) \wedge \omega_0^1 = 0,$$

which by Cartan's lemma implies that

$$db_2 - 3b_2\omega_1^1 = b_3\omega_0^1. \tag{1.101}$$

If we fix A_0 on Γ (i.e., if we set $\omega_0^1 = 0$), we find that

$$\delta b_2 - 3b_2\pi_1^1 = 0, \tag{1.102}$$

where $\pi_i^j = \omega_i^j|_{\omega_0^1=0}$ and δ is the symbol differentiation with respect to the secondary parameters (i.e., the parameters that move a frame when the point $x = A_0$ is held fixed).

By (1.100), it follows from (1.102) that

$$\delta \log b_2 = 3\pi_1^1. \tag{1.103}$$

If we fix all secondary parameters except ϕ_2, in terms of the differential of which the secondary form $3\pi_1^1$ is expressed, we obtain

$$3\pi_1^1 = \delta \log \phi_2. \tag{1.104}$$

Here we used the fact that the differential of a function of one variable is always a total differential. By (1.104), equation (1.103) takes the form

$$\delta \log b_2 = \delta \log \phi_2.$$

It follows that

$$b_2 = E_2 \phi_2,$$

where $E_2 = \text{const}$. Because ϕ_2 takes arbitrary values, we can take

$$\phi_2 = \frac{1}{E_2}.$$

As a result, we get

$$b_2 = 1, \tag{1.105}$$

and (1.99) takes the form

$$\omega_1^2 = \omega_0^1. \tag{1.106}$$

Note that we could take $\phi_2 = -\frac{1}{E_2}$, and as a result we could have $b_2 = -1$ and

$$\omega_1^2 = -\omega_0^1. \tag{1.107}$$

Note that if we change the orientation of the curve Γ, i.e., if we change du to $-du$, we come again to equations (1.105) and (1.106). In what follows, we will assume that specialization (1.105) takes place.

By (1.105), equation (1.101) takes the form

$$-3\omega_1^1 = b_3 \omega_0^1. \tag{1.108}$$

Taking the exterior derivative of equation (1.108), we obtain the exterior quadratic equation

$$[db_3 + b_3(\omega_0^0 - \omega_1^1) + 3(\omega_1^0 - \omega_2^1)] \wedge \omega_0^1 = 0,$$

which by Cartan's lemma implies that

$$db_3 + b_3(\omega_0^0 - \omega_1^1) + 3(\omega_1^0 - \omega_2^1) = (3b_4 - (b_3)^2)\omega_0^1. \tag{1.109}$$

1.4.2 Power Series Expansion of an Equation of a Curve. Before going to further frame specializations, we will clarify the meaning of the functions b_2, b_3, and b_4. First, we find the conditions for a point

$$M = x^0 A_0 + x^1 A_1 + x^2 A_2 \tag{1.110}$$

in the plane $\mathbb{P}^2$ to be fixed. Such a condition is

$$dM = \theta M, \tag{1.111}$$

where θ is a 1-form. By (1.93) and linear independence of the vertices A_u of our moving frame, it follows from (1.111) that

$$\left\{ \begin{array}{l} dx^0 + x^0\omega_0^0 + x^1\omega_1^0 + x^2\omega_2^0 = \theta x^0, \\ dx^1 + x^0\omega_0^1 + x^1\omega_1^1 + x^2\omega_2^1 = \theta x^1, \\ dx^2 + x^0\omega_0^2 + x^1\omega_1^2 + x^2\omega_2^2 = \theta x^2. \end{array} \right. \tag{1.112}$$

For nonhomogeneous coordinates

$$x = \frac{x^1}{x^0}, \quad y = \frac{x^2}{x^0}$$

of the point M, equations (1.112) give

$$\left\{ \begin{array}{l} dx + \omega_0^1 + x(\omega_1^1 - \omega_0^0) + y\omega_2^1 - x^2\omega_1^0 - xy\omega_2^0 = 0, \\ dy + \omega_0^2 + x\omega_1^2 + y(\omega_2^2 - \omega_0^0) - xy\omega_1^0 - y^2\omega_2^0 = 0. \end{array} \right. \tag{1.113}$$

Suppose that the curve Γ is given by an equation

$$y = f(x). \tag{1.114}$$

If we place the origin $(0, 0)$ to a regular point of Γ, then the right-hand side of (1.114) can be expanded into the MacLauren series:

$$y = a_1 x + \frac{1}{2!}a_2 x^2 + \frac{1}{3!}a_3 x^3 + \ldots = \sum_{n=1}^{\infty} \frac{1}{n!} a_n x^n. \tag{1.115}$$

Because we placed the point A_1 on the tangent $T_x(\Gamma)$ to Γ at the point x, the equation of the tangent line $T_x(\Gamma)$ is $y = 0$. The tangent line $T_x(\Gamma)$ intersects Γ in two coinciding points. Thus if we set $y = 0$ in (1.115), we must obtain a double root $x = 0$. Hence expansion (1.115) must start from second-degree terms. Therefore, we have

$$a_1 = 0, \tag{1.116}$$

and expansion (1.115) becomes

$$y = \frac{1}{2!}a_2 x^2 + \frac{1}{3!}a_3 x^3 + \ldots = \sum_{n=2}^{\infty} \frac{1}{n!} a_n x^n. \tag{1.117}$$

Next we differentiate equation (1.117), apply (1.113), and equate the coefficients in x. This gives

$$a_2 = 1. \tag{1.118}$$

Equating the coefficients in $x^n, n \geq 2$, we obtain the following recurrent differential equations for the coefficients a_n, $n \geq 2$:

$$\begin{aligned} da_n &+ a_n[(n-1)\omega_0^0 - n\omega_1^1 + \omega_2^2] + n(n-2)a_{n-1}\omega_1^0 \\ &+ n!(s_2^0\omega_2^0 - s_2^1\omega_2^1) = a_{n+1}\omega_0^1, \end{aligned} \tag{1.119}$$

where

$$\begin{cases} s_2^0 = \displaystyle\sum_{\alpha,\beta} \frac{\alpha-1}{\alpha!\beta!} a_\alpha a_\beta, \quad \alpha+\beta = n, \quad \alpha > 1, \quad n > 2, \\ s_2^1 = \displaystyle\sum_{\alpha,\beta} \frac{\alpha}{\alpha!\beta!} a_\alpha a_\beta, \quad \alpha+\beta = n+1. \end{cases} \tag{1.120}$$

Substituting $n = 2, 3$ into (1.119) and applying (1.119), (1.116), (1.118), (1.96), (1.106), (1.108), and (1.109), we find that

$$a_3 = b_3, \quad a_4 = 3b_4. \tag{1.121}$$

As a result of (1.118) and (1.121), expansion (1.117) takes the form

$$y = \frac{1}{2}x^2 + \frac{1}{6}b_3x^3 + \frac{1}{8}b_4x^4 + [5]. \tag{1.122}$$

Hence, equation (1.122) shows that *the coefficients* a_2, a_3*, and* a_4 *of expansion* (1.117) *coincide with the functions* $b_2 = 1, b_3$*, and* b_4.

1.4.3 The Osculating Conic to a Curve. In homogeneous coordinates, the equation of a conic in the plane $\mathbb{P}^2$ is

$$a_{11}(x^1)^2 + 2a_{12}x^1x^2 + a_{22}(x^2)^2 + 2a_{10}x^1x^0 + 2a_{20}x^2x^0 + a_{00}(x^0)^2 = 0.$$

It has six coefficients but only five essential parameters. Thus a conic can have a fourth-order tangency with the curve Γ. We write the preceding equation in nonhomogeneous coordinates x and y as follows:

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{20}y + a_{00} = 0. \tag{1.123}$$

If (1.123) is the equation of such an osculating conic C_2, then substituting y from (1.122) into (1.123), we must obtain five roots $x = 0$. Thus the coefficients in $x^k, k = 0, 1, 2, 3, 4$, must vanish. Hence we obtain five relations between a_{uv} and the coefficients of expansion (1.122). But we can obtain the same five relations by another method. The function y and its derivatives $y', y'', y''', y^{(iv)}$ computed from equations (1.122) and (1.123) must coincide at the point $x = 0$, $y = 0$. Thus, taking four consecutive derivatives of (1.123) and substituting each time the values

$$x = 0,\ y = 0,\ y'(0) = 0,\ y''(0) = 1,\ y'''(0) = b_3,\ y^{(iv)}(0) = 3b_4,$$

we obtain the following relations:

$$\begin{cases} a_{00} = 0, \ \ a_{10} = 0, \ \ a_{11} + a_{20} = 0, \ \ 3a_{12} + a_{20}b_3 = 0, \\ 4a_{12}b_3 + 3a_{22} + 3a_{20}b_4 = 0. \end{cases} \tag{1.124}$$

Solving (1.124) with respect to a_{11}, a_{12}, and a_{22}, we find that

$$\begin{cases} a_{00} = 0, \ \ a_{10} = 0, \ \ a_{11} = -a_{20}, \\ a_{12} = -\dfrac{1}{3}b_3 a_{20}, \ \ a_{22} = \dfrac{1}{9}\Big(4(b_3)^2 - 9b_4\Big)a_{20}. \end{cases} \tag{1.125}$$

Substituting (1.125) into (1.123), we obtain the following equation (in nonhomogeneous coordinates) of the osculating conic C_2 having a fourth-order tangency with the curve Γ:

$$9x^2 + 6b_3xy + (9b_4 - 4(b_3)^2)y^2 - 18y = 0. \tag{1.126}$$

In homogeneous coordinates (x^0, x^1, x^2), equation (1.126) can be written as

$$9(x^1)^2 + 6b_3x^1x^2 + (9b_4 - 4(b_3)^2)(x^2)^2 - 18x^2x^0 = 0. \tag{1.127}$$

1.4.4 The Second and Third Specializations and Their Geometric Meaning. For the next two specializations, we apply the geometric method. First, we place the point $A_2(0, 0, 1)$ on the conic C_2. The point A_2 belongs to the conic C_2 defined by equation (1.127) if and only if

$$9b_4 - 4(b_3)^2 = 0. \tag{1.128}$$

As a result of this specialization, equation (1.127) takes the form

$$3(x^1)^2 + 2b_3x^1x^2 - 6x^2x^0 = 0. \tag{1.129}$$

Second, we locate the point A_2 on the tangent line to C_2 at the point A_2. The equation of the tangent to the curve (1.129) at A_2 is

$$\frac{\partial F}{\partial x^2} = 2b_3x^1 - 6x^0 = 0$$

or

$$b_3x^1 - 3x^0 = 0.$$

This tangent line is the line $A_1 \wedge A_2$ defined by the equation $x^0 = 0$ if and only if

$$b_3 = 0. \tag{1.130}$$

It follows from (1.128) and (1.130) that *the conditions*

$$b_3 = 0, \quad b_4 = 0 \tag{1.131}$$

are necessary and sufficient conditions for A_2 to be located on the curve Γ and for the line $A_1 \wedge A_2$ to be the tangent line to Γ at the point A_2.

The specializations (1.131) imply that equations (1.108) and (1.109) become

$$\omega_1^1 = 0 \tag{1.132}$$

and

$$\omega_1^0 - \omega_2^1 = 0. \tag{1.133}$$

In addition, it follows from (1.96) and (1.132) that

$$\omega_0^0 + \omega_2^2 = 0. \tag{1.134}$$

Next, taking the exterior derivative of equation (1.133), we obtain the exterior quadratic equation

$$\omega_2^0 \wedge \omega_0^1 = 0,$$

which by Cartan's lemma implies that

$$\omega_2^0 = b_5\omega_0^1. \tag{1.135}$$

In addition, as a result of specializations (1.131), expansion (1.122) takes the form

$$y = \frac{1}{2}x^2 + [5], \tag{1.136}$$

and equation (1.126) of the osculating conic C_2 becomes

$$y = \frac{1}{2}x^2. \tag{1.137}$$

Next, we rewrite expansion (1.122) in the form

$$y = \frac{1}{2}x^2 + \frac{1}{5!}a_5x^5 + \frac{1}{6!}a_6x^6 + \frac{1}{7!}a_7x^7 + [8]. \tag{1.138}$$

If $a_5 = 0$, then at the point $x = A_0$, the conic C_2 has a tangency of at least fifth order with the curve Γ. Such points of Γ are called *sextactic*. If Γ and C_2 have a fifth-order tangency at all points, then C_2 is the same at all points of Γ, and $\Gamma = C_2$.

It is easy to confirm this consideration analytically. If $a_5 = 0$, then it follows from (1.119)–(1.120) that $a_p = 0, p = 6, 7, \ldots$, and expansion (1.138) becomes (1.137).

In what follows, we assume that

$$a_5 \neq 0. \tag{1.139}$$

Setting $n = 4$ in (1.119)–(1.120) and taking into account (1.131) and (1.135), we find that

$$a_5 = 6b_5. \tag{1.140}$$

By (1.140), expansion (1.138) takes the form

$$y = \frac{1}{2}x^2 + \frac{1}{20}b_5x^5 + \frac{1}{6!}a_6x^6 + \frac{1}{7!}a_7x^7 + [8]. \tag{1.141}$$

It follows from (1.139) and (1.140) that

$$b_5 \neq 0. \tag{1.142}$$

In what follows, we assume that (1.139) (or (1.142)), i.e., that *the curve* Γ *in question is not a conic.* Note that in the frame we have constructed, the conic C_2 defined by equation (1.137) has a fourth-order tangency with the curve Γ at the point $x = A_0$.

1.4.5 The Osculating Cubic to a Curve. In homogeneous coordinates, the equation of a cubic in the plane $\mathbb{P}^2$ is

$$a_{uvw}x^ux^vx^w = 0, \quad u, v, w = 0, 1, 2. \tag{1.143}$$

It has ten coefficients but only nine essential parameters. Thus a cubic can have an eighth-order tangency with the curve Γ. But not all cubics in $\mathbb{P}^2$ are projectively equivalent. Moreover, not all points of a cubic are equivalent: a cubic can have a singular (double) point. We will use this fact later. Thus we will save one parameter and look for a cubic C_3 having only a seven-order tangency with the curve Γ.

In order to find such a cubic C_3, we apply the same procedure we used in Section 1.4.3 when we were looking for an osculating conic C_2. The function y and its derivatives $y', y'', y''', y^{(iv)}, y^{(v)}, y^{(vi)}, y^{(vii)}$ computed from equations (1.138) (we assume that the specializations $b_2 = 1, b_3 = b_4 = 0$ have been made) and (1.143) must coincide at the point $x = 0$, $y = 0$. Thus taking seven consecutive derivatives of (1.143) and substituting each time the values

$$\begin{array}{lllll} x = 0, & y = 0, & y'(0) = 0, & y''(0) = 1, & y'''(0) = 0, \\ y^{(iv)}(0) = 0, & y^{(v)} = a_5, & y^{(vi)} = a_6, & y^{(vii)} = a_7, & \end{array}$$

we find the following eight relations:

$$\begin{aligned} & a_{000} = 0, \;\; a_{100} = 0, \\ & 2a_{110} + a_{200} = 0, \;\; a_{111} + 3a_{120} = 0, \\ & a_{220} + 2a_{112} = 0, \;\; a_5 a_{200} + 30a_{122} = 0, \\ & a_6 a_{200} + 12a_5 a_{120} + 30a_{222} = 0, \\ & a_7 a_{200} + 14a_6 a_{120} + 42a_5(a_{112} + a_{220}) = 0. \end{aligned}$$

Excluding the case when Γ is a conic (i.e., assuming that inequality (1.139) holds), we find from the preceding equations that

$$\begin{aligned} & a_{110} = \lambda a_5, \;\; a_{120} = \mu a_5, \;\; a_{200} = -2\lambda a_5, \;\; a_{111} = -3\mu a_5, \\ & 3a_{122} = \frac{1}{5}\lambda a_5^2, \;\; a_{222} = \frac{1}{15}\lambda a_5 a_6 - \frac{2}{5} a_5^2 \mu, \\ & 3a_{112} = -\frac{1}{7}\lambda a_7 + \mu a_6, \;\; 3a_{220} = -\frac{2}{7}\lambda a_7 + 2\mu a_6, \end{aligned}$$

where λ and μ are arbitrary parameters.

As a result, we find the following equation of a pencil of osculating cubics having at the point $x = y = 0$ a seven-order tangency with the curve Γ:

$$\begin{aligned} & \lambda\left[a_5\left(3x^2 - 6y + \frac{a_5}{5}xy^2 + \frac{a_6}{15}y^3\right) + \frac{a_7}{7}\left(2y^2 - x^2 y\right)\right] \\ & + \mu\left[a_5\left(6xy - 3x^3 - \frac{2a_5}{5}y^3\right) - a_6\left(2y^2 - x^2 y\right)\right] = 0. \end{aligned} \tag{1.144}$$

We can see that if $\lambda = 0$, equation (1.144) does not contain the first powers of x and y. Hence if $\lambda = 0$, the cubic C_3 has a double point (a knot) at the point A_0. Equating to zero the second-degree terms, we get

$$6a_5 xy - 2a_6 y^2 = 0.$$

This shows that at the double point A_0, the osculating conic C_3 has two real tangents defined by the equations

$$y = 0 \text{ and } 3a_5 x - a_6 y = 0. \tag{1.145}$$

1.4.6 Two More Specializations and Their Geometric Meaning. For the two final specializations, we will apply again the analytic method. Taking the exterior derivative of equation (1.135), we obtain the exterior quadratic equation

$$(db_5 + 3b_5\omega_0^0) \wedge \omega_0^1 = 0,$$

which by Cartan's lemma implies that

$$db_5 + 3b_5\omega_0^0 = 3b_6\omega_0^1. \tag{1.146}$$

If the point $x = A_0$ is held fixed, it follows from (1.146) and (1.141) that

$$\delta \log b_5 = -3\pi_0^0.$$

Fixing all secondary parameters except ϕ_5, in terms of the differential of which the secondary form $-3\pi_0^0$ is expressed, we obtain consecutively

$$-3\pi_0^0 = \delta \log \phi_5, \quad \delta \log b_5 = \delta \log \phi_5, \quad b_5 = E_5\phi_5,$$

where $E_5 = \text{const}$. Taking $\phi_5 = \frac{1}{E_5}$, we arrive at

$$b_5 = 1. \tag{1.147}$$

By (1.147), we find from (1.146) that

$$\omega_0^0 = b_6\omega_0^1. \tag{1.148}$$

Taking the exterior derivative of equation (1.148), we obtain the exterior quadratic equation

$$(db_6 + \omega_1^0) \wedge \omega_0^1 = 0,$$

which by Cartan's lemma implies that

$$db_6 + \omega_1^0 = k\omega_0^1. \tag{1.149}$$

If the point $x = A_0$ is held fixed, it follows from (1.149) that

$$\delta b_6 = -\pi_1^0.$$

Fixing all secondary parameters except the parameter ϕ_6, in terms of the differential of which the secondary form $-\pi_1^0$ is expressed, we obtain consecutively

$$-\pi_1^0 = \delta\phi_6, \quad \delta b_6 = \delta\phi_6, \quad b_6 = \phi_6 + E_6,$$

where $E_6 = \text{const}$. Taking $\phi_6 = -E_6$, we arrive at

$$b_6 = 0. \tag{1.150}$$

By (1.150), equation (1.149) becomes

$$\omega_1^0 = k\omega_0^1, \tag{1.151}$$

and equations (1.148) and (1.134) give

$$\omega_0^0 = \omega_2^2 = 0. \tag{1.152}$$

Exterior differentiation of (1.151) gives the exterior quadratic equation

$$dk \wedge \omega_0^1 = 0,$$

which by Cartan's lemma implies that

$$dk = l\omega_0^1. \tag{1.153}$$

Now all the forms ω_i^j become the principal forms:

$$\begin{cases} \omega_0^2 = 0, & \omega_1^2 = \omega_2^0 = \omega_0^1, \\ \omega_1^0 = \omega_2^1 = k\omega_0^1, & \omega_0^0 = \omega_1^1 = \omega_2^2 = 0, \end{cases} \tag{1.154}$$

and the functions k and l are the absolute invariants of the curve Γ.

Setting $n = 5$ and $n = 6$ in (1.119)–(1.120) and taking into account (1.131), (1.140), (1.147), (1.150), (1.151), and (1.152), we find that

$$a_6 = 0, \quad a_7 = 18k. \tag{1.155}$$

By (1.155), expansion (1.141) takes the form

$$y = \frac{1}{2}x^2 + \frac{1}{20}x^5 + \frac{k}{280}x^7 + [8]. \tag{1.156}$$

The osculating cubic C_3 having the knot at the origin is determined by the equation

$$5x^3 + 4y^3 - 10xy = 0. \tag{1.157}$$

Note that expansion (1.156) coincides with the similar decomposition (8) on p. 216 in the book [Fi 37] by Finikov.

Now we can establish the geometric meaning of the specialization $a_6 = 0$. It follows from (1.145), that *the condition* $a_6 = 0$ *is necessary and sufficient* *for the straight line A_0A_2 to be the second tangent line to the osculating cubic C_3 at its double point A_0.*

Note that we can make the specialization $a_6 = 0$ geometrically by requesting the line A_0A_2 be the second tangent line to the cubic C_3 at its double point A_0 immediately after we found equations (1.145).

1.4.7 Conclusions. We make the following conclusions from our considerations in this section:

1. *The specializations we have performed can be made for any curve not a straight line or a conic.*

2. We summarize here the geometric meaning of all vertices of our specialized moving frame:

$$\begin{array}{lll} A_0 = x \in \Gamma, & A_0 \in C_2, & A_0 \in C_3, \\ A_1 \in T_x(\Gamma) & A_1 \in T_x(C_3), & A_2 \in T_x(C_2) \\ A_2 = T_x(C_3) \cap C_2, & A_1 = T_x(\Gamma) \cap T_{A_2}(C_2) & \end{array} \tag{1.158}$$

(see Figure 1.1).

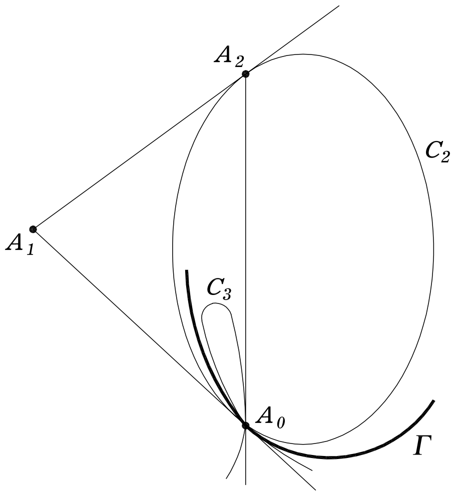

Figure 1.1

3. By (1.94) and (1.154), we have

$$d\omega_0^1 = 0,$$

i.e., the basic form ω_0^1 is a total differential:

$$\omega_0^1 = du. \tag{1.159}$$

By (1.93), (1.154), and (1.159), we obtain the following Frenet formulas:

$$\begin{cases} \dfrac{dA_0}{du} = A_1, \\ \dfrac{dA_1}{du} = kA_0 + A_1, \\ \dfrac{dA_2}{du} = A_0 + kA_1, \end{cases} \tag{1.160}$$

The parameter u in (1.159) is the *projective arc length*, and the absolute invariant k in equations (1.160) is the *projective curvature of* Γ (see more detail on the projective arc length on pp. 222–224 and on the projective curvature on pp. 221–222 and 225–226 in the book [Fi 37] by Finikov).

It can be proved (see [Fi 37], pp. 220–221) that if we take the new parameter v such that $\frac{dx}{dv} = -1$, then

$$dv = \sqrt[3]{\left(A_0, \frac{dA_0}{du}, \frac{d^2A_0}{du^2}\right)}\, du$$

and

$$k = -\frac{1}{2}\left(A_0, \frac{d^2A_0}{dv^2}, \frac{d^3A_0}{dv^3}\right),$$

and if the projective curvature k is given as a function of the projective arc length, then the curve Γ is defined up to a projective transformation. In particular, in the book [Fi 37], the curves Γ with $k = 0$ and $k = \text{const}$ are determined.

In general, all these considerations are coming from Halphen's paper [H 78]. In particular, Halphen defined the so-called Halphen's point in the following manner (see also p. 68 in the book [Wi 06] by Wilczynski).

A pencil of cubics has always $3 \cdot 3 = 9$ centers. All curves of the pencil (1.144) of osculating cubics have eight common points with the curve Γ (and thus among each other) at the origin. Therefore, besides the point $x = A_0$, there exists only one common points for all cubics of the pencil (1.144). This point is the *Halphen point*. Its coordinates are

$$x_1 = 490k, \ \ x_2 = 175k^2, \ \ x_0 = 685 + 25k^2.$$

If $k = 0$, then the Halphen point coincides with the point $x = A_0$. In this case the curve (1.157) has an eighth-order tangency with Γ at x.

1.5 Some Algebraic Manifolds

1.5.1 Grassmannians. We now consider some algebraic varieties in a projective space, which we will need in our considerations.

First of all, we study the Grassmannian $\mathbb{G}(m, n)$ of m-dimensional subspaces in a projective space $\mathbb{P}^n$. Consider a fixed frame $\{E_u\}$ in $\mathbb{P}^n$ and denote the coordinates of a point X relative to this frame by x^u. Thus, we have $X = x^u E_u$. Let $\mathbb{P}^m$ be an m-dimensional subspace in $\mathbb{P}^n$. Let us take $m+1$ linearly independent points $X_i, \ \ i = 0, 1, \ldots, m$, in the subspace $\mathbb{P}^m$. We call them *basis points* of the $\mathbb{P}^m$. We write the coordinates of the points X_i relative to the frame $\{E_u\}$ in the form of a matrix:

$$(x_i^u) = \begin{pmatrix} x_0^0 & x_0^1 & \ldots & x_0^n \\ x_1^0 & x_1^1 & \ldots & x_1^n \\ \multicolumn{4}{c}{\dotfill} \\ x_m^0 & x_m^1 & \ldots & x_m^n \end{pmatrix}. \tag{1.161}$$

Consider the minors $p^{i_0 i_1 \ldots i_m}$ of order $m+1$ of this matrix:

$$p^{i_0 i_1 \ldots i_m} = \det \begin{pmatrix} x_0^{i_0} & x_0^{i_1} & \ldots & x_0^{i_m} \\ x_1^{i_0} & x_1^{i_1} & \ldots & x_1^{i_m} \\ \multicolumn{4}{c}{\dotfill} \\ x_m^{i_0} & x_m^{i_1} & \ldots & x_m^{i_m} \end{pmatrix}. \tag{1.162}$$

Because the matrix has $m+1$ rows and $n+1$ columns, the total number of such minors is equal to $\binom{n+1}{m+1}$. If we change the basis in the subspace $\mathbb{P}^m$, the matrix (1.161) also changes, but all of its minors are multiplied by the same factor, namely, the determinant of the matrix of basis transformation. Thus, these minors can be taken as homogeneous projective coordinates of a point in the projective space $\mathbb{P}^N$ of dimension $N = \binom{n+1}{m+1} - 1$. These coordinates are called the *Grassmann coordinates* of the $\mathbb{P}^m \subset \mathbb{P}^n$. It is easy to see that these coordinates are skew-symmetric and that not proportional sets of Grassmann coordinates correspond to different m-dimensional subspaces.

The Grassmann coordinates $p^{i_0 i_1 \dots i_m}$ are not independent—they satisfy the sequence of the following quadratic relations:

$$p^{i_0 i_1 \dots i_{m-1}[i_m} p^{j_0 j_1 \dots j_m]} = 0, \tag{1.163}$$

which follows from equations (1.161) and (1.162) (see, for example, Hodge and Pedoe [HP 47]). In formulas (1.163) (and many other formulas of this book), the square brackets enclosing some (or all) upper (or lower) indices denote the alternation with respect to the enclosed indices while the parentheses in the indices denote the symmetrization. For example,

$$\begin{aligned} t^{[ij]} &= \frac{1}{2}(t^{ij} - t^{ji}), \ \ t^{(ij)} = \frac{1}{2}(t^{ij} + t^{ji}), \\ t^{[ijk]} &= \frac{1}{3!}(t^{ijk} + t^{jki} + t^{kij} - t^{jik} - t^{kji} - t^{ikj}), \\ t^{(ijk)} &= \frac{1}{3!}(t^{ijk} + t^{jki} + t^{kij} + t^{jik} + t^{kji} + t^{ikj}). \end{aligned}$$

If we locate points A_i, $i = 0, 1, \dots, m$, of the moving frame in the subspace $\mathbb{P}^m$, then we have

$$dA_i = \omega_i^j A_j + \omega_i^\alpha A_\alpha, \quad \alpha = m+1, \dots, n;$$

thus the 1-forms ω_i^α are basis forms on the Grassmannian $\mathbb{G}(m, n)$.

Relations (1.163) define in the space $\mathbb{P}^N$ an algebraic variety of dimension $(m+1)(n-m)$, which is the number of linearly independent basis forms ω_i^α on the Grassmannian. We denote this algebraic variety by $\Omega(m, n)$. There is a one-to-one correspondence between m-dimensional subspaces $\mathbb{P}^m$ of $\mathbb{P}^n$ and the points of the variety $\Omega(m, n)$. This correspondence defines the mapping $\varphi : \mathbb{G}(m, n) \to \Omega(m, n)$, called the *Grassmann mapping*.

As an example, we consider the Grassmannian $\mathbb{G}(1, 3)$, the manifold of straight lines of the three-dimensional projective space $\mathbb{P}^3$. In this case, matrix (1.161) takes the form:

$$\begin{pmatrix} x_0^0 & x_0^1 & x_0^2 & x_0^3 \\ x_1^0 & x_1^1 & x_1^2 & x_1^3 \end{pmatrix}.$$

Its minors

$$p^{i_0 i_1} = \begin{vmatrix} x_0^{i_0} & x_0^{i_1} \\ x_1^{i_0} & x_1^{i_1} \end{vmatrix}$$

are usually called the *Plücker coordinates* of the straight line l defined by the points X_0 and X_1. Because $\binom{4}{2} = 6$, the minors are homogeneous projective coordinates of a point in the space $\mathbb{P}^5$. It is easy to prove that these coordinates satisfy the single quadratic equation

$$p^{01}p^{23} + p^{02}p^{31} + p^{03}p^{12} = 0$$

(cf. (1.163)). Therefore, the variety $\Omega(1, 3)$ is a hyperquadric in $\mathbb{P}^5$, called the *Plücker hyperquadric.*

Let us study the structure of the Grassmannian $\mathbb{G}(m, n)$ and its image $\Omega(m, n)$ in the space $\mathbb{P}^N$, where $N = \binom{n+1}{m+1} - 1$. Let p and q be two m-dimensional subspaces of $\mathbb{P}^n$ having in common an $(m-1)$-dimensional subspace $\mathbb{P}^{m-1}$. These two subspaces generate a linear pencil $\lambda p + \mu q$ of m-dimensional subspaces. A straight line of the variety $\Omega(m, n)$ corresponds to this pencil. All subspaces of the pencil belong to the same subspace $\mathbb{P}^{m+1}$ of dimension $m+1$, and a pair of subspaces $\mathbb{P}^{m-1} \subset \mathbb{P}^{m+1}$ completely defines the pencil and therefore a straight line on $\Omega(m, n)$.

Consider further an $(n-m)$-bundle of m-dimensional subspaces passing through a fixed subspace $\mathbb{P}^{m-1}$. An $(n-m)$-dimensional plane generator ξ^{n-m} of the variety $\Omega(m, n)$ corresponds to this bundle. Because the space $\mathbb{P}^n$ contains the $m(n-m+1)$-dimensional family of subspaces $\mathbb{P}^{m-1}$, the variety $\Omega(m, n)$ carries a family of $(n-m)$-dimensional plane generators ξ^{n-m}, and the latter family depends on $m(n-m+1)$ parameters.

Let $\mathbb{P}^{m+1}$ be a fixed $(m+1)$-dimensional subspace in $\mathbb{P}^n$. Consider all its m-dimensional subspaces $\mathbb{P}^m$. They form a plane field of dimension $m+1$. An $(m+1)$-dimensional plane generator η^{m+1} of the variety $\Omega(m, n)$ corresponds to this field. Because $\mathbb{P}^n$ contains the $(m+2)(n-m-1)$-parameter family of subspaces $\mathbb{P}^{m+1}$, the variety $\Omega(m, n)$ carries an $(m+2)(n-m-1)$-parameter family of plane generators η^{m+1}.

If $\mathbb{P}^{m-1} \subset \mathbb{P}^{m+1}$, then the plane generators ξ^{n-m} and η^{m+1} of the variety $\Omega(m, n)$ corresponding to these subspaces intersect each other along a straight line. Otherwise, they do not have common points.

Next, consider in $\mathbb{P}^n$ a fixed subspace $\mathbb{P}^m$. It contains an m-parameter family of subspaces $\mathbb{P}^{m-1}$. Thus, an m-parameter family of generators ξ^{n-m} passes through the point $p \in \Omega(m, n)$ corresponding to the $\mathbb{P}^m$. There is also an $(n-m-1)$-parameter family of subspaces $\mathbb{P}^{m+1}$ passing through the same subspace $\mathbb{P}^m$. Thus, an $(n-m-1)$-parameter family of generators η^{m+1} passes through the point $p \in \Omega(m, n)$. Moreover, any two generators ξ^{n-m} and η^{m+1} passing through the point p have a straight line as their intersection. It follows that all plane generators ξ^{n-m} and η^{m+1} passing through the point $p \in \Omega(m, n)$ are generators of a cone with its vertex at the point p, and this cone is located on the variety $\Omega(m,n)$. We denote this cone by $C_p(n-m,m+1)$ and call it the *Segre cone.* The projectivization of the Segre cone with the center at a point p is the *Segre variety* $S(n-m-1,m)$ which we will study later.

In the space $\mathbb{P}^n$, the set of all m-dimensional subspaces intersecting a fixed subspace $\mathbb{P}^m$ along the subspace of dimension $m-1$ corresponds to the Segre cone $C_p(n-m,m+1)$. It follows that the dimension of the Segre cone $C_p(n-m,m+1)$ is equal to n.

1.5.2 Determinant Submanifolds. The so-called determinant submanifolds are interesting examples of submanifolds in a projective space.

Consider a projective space $\mathbb{P}^N$ of dimension $N = ml + m + l$ in which projective coordinates are matrices (x_i^α) with $i = 0, 1, \ldots, m;\ \ \alpha = 0, 1, \ldots, l$, and we suppose $m \leq l$. A *determinant manifold* is defined by the condition

$$1 \leq \operatorname{rank}\,(x_i^\alpha) \leq r, \ \ r \leq m. \tag{1.164}$$

Consider first the extreme case $r = 1$. In this case, the matrix (x_i^α) has the form of a simple dyad:

$$x_i^\alpha = t^\alpha s_i, \tag{1.165}$$

where t^α and s_i are homogeneous parameters that can be taken as coordinates of points in the spaces $\mathbb{P}^l$ and $\mathbb{P}^{m*}$.

The determinant manifold defined by equation (1.165) is called the *Segre variety* and is denoted by $S(m,l)$ (cf. the end of Section 1.4.1). This variety carries two families of plane generators $s_i = \lambda c_i$ and $t^\alpha = \mu c^\alpha$ where c_i and c^α are constants. The generators of these two families are of dimension l and m, respectively. The Segre variety is an embedding

$$\mathbb{P}^l \times (\mathbb{P}^m)^* \to \mathbb{P}^N \tag{1.166}$$

of the direct product of the spaces $\mathbb{P}^l$ and $\mathbb{P}^{m*}$ into the space $\mathbb{P}^N$, and the dimension of the Segre variety is $l + m$.

Suppose now that in relation (1.164) the rank $r = 2$. In this case the entries of the matrix (x_i^α) can be written in the form

$$x_i^\alpha = \lambda\ \ ({}'t^\alpha \cdot s'_i) + \mu\ \ ({}''t^\alpha \cdot s''_i), \tag{1.167}$$

i.e., the matrix (x_i^α) is a linear combination of two simple dyads. Each of these dyads determines a point on the Segre variety $S(m,l)$. If the parameters λ and μ vary, the point of the space $\mathbb{P}^N$ with coordinates x_i^α describes a straight line —the *bisecant* of the Segre variety $S(m,l)$. Thus, if $r = 2$, equation (1.164) defines the *bisecant variety* for the Segre variety $S(m,l)$.

Similarly, for any r, the determinant manifold (1.164) is a family of $(r-1)$-secant subspaces for the Segre variety $S(m,l)$.

For example, if $m = l = 1$, then $n = 3$, and the equations of the Segre variety $S(1,1)$ can be written as

$$\begin{aligned} x_0^0 &= t^0 s_0, \quad x_1^0 = t^0 s_1, \\ x_0^1 &= t^1 s_0, \quad x_1^1 = t^1 s_1. \end{aligned} \tag{1.168}$$

Eliminating the parameters t^α and s_i from these equations, we arrive at the quadratic equation

$$x_0^0 x_1^1 - x_0^1 x_1^0 = 0, \tag{1.169}$$

defining in the space $\mathbb{P}^3$ a ruled surface of second order that carries two families of rectilinear generators: $s_i = \text{const}$ and $t^\alpha = \text{const}$. This surface is an embedding of the direct product $\mathbb{P}^1 \times \mathbb{P}^{1*}$ into the space $\mathbb{P}^3$.

Next, we consider another type of determinant manifold defined in a projective space $\mathbb{P}^n$ of dimension $n = \frac{1}{2}(m+1)(m+2) - 1$, where projective coordinates are symmetric matrices (x^{ij}), $i, j = 0, 1, \ldots, m$, by the equation:

$$\operatorname{rank}\,(x^{ij}) = r, \quad r \leq m. \tag{1.170}$$

If $r = 1$, then each entry of a matrix (x^{ij}) is the tensorial square of a vector t^i:

$$x^{ij} = t^i t^j. \tag{1.171}$$

The parameters t^i can be considered as homogeneous coordinates of a point in a projective space $\mathbb{P}^m$. Thus, the manifold defined by equations (1.171) is a symmetric embedding of the $\mathbb{P}^m$ into the $\mathbb{P}^n$:

$$s: \ \operatorname{Sym}\,(\mathbb{P}^m \times \mathbb{P}^m) \to \mathbb{P}^n,$$

where $n = \frac{1}{2}(m+1)(m+2) - 1$. The manifold (1.171) is called the *Veronese variety* and is denoted by $V(m)$. Its dimension is m.

If $r > 1$, the determinant manifold (1.170) is the variety of $(r-1)$-secant subspaces for the Veronese variety $V(m)$.

As an example of the Veronese variety, we consider the case $m = 2$. Then $n = 5$ and the variety $V(2)$, defined by equation (1.171) for $i, j = 0, 1, 2$, is a symmetric embedding of the two-dimensional projective plane into the space $\mathbb{P}^5$. The variety $V(2)$ is a two-dimensional surface of fourth order in $\mathbb{P}^5$ (see, for example, Semple and Roth [SR 85]).

Note some properties of the Veronese surface $V(2)$. To each straight line of the plane $\mathbb{P}^2$ there corresponds a conic on the Veronese surface $V(2)$, and this surface carries a two-parameter family of such conics. Through each point of the surface $V(2)$, there passes a one-parameter family of such conics, and through any pair of points of the surface $V(2)$ there passes a unique conic of this family. Two-dimensional planes in $\mathbb{P}^5$ containing these conics are called *conisecant planes* of the surface $V(2)$.

To the conics defined by the equation

$$a_{ij}t^i t^j = 0 \tag{1.172}$$

in the plane $\mathbb{P}^2$, there corresponds a quartic (a fourth-degree curve) on the Veronese surface $V(2)$. This quartic is the intersection of the Veronese surface $V(2)$ with the hyperplane

$$a_{ij}x^{ij} = 0 \tag{1.173}$$

of the space $\mathbb{P}^5$. If the conic (1.172) degenerates into two straight lines, then the corresponding quartic is decomposed into two conics. For curves of this type, we have $\det(a_{ij}) = 0$, and the hyperplane (1.173) defining this curve is tangent to the Veronese surface $V(2)$ at a point of intersection of these two conics. If the curve (1.172) is a double straight line, then $a_{ij} = a_i a_j$, and the hyperplane (1.173) is tangent to the Veronese surface $V(2)$ along a double conic.

If $r = 2$, the manifold defined in the space $\mathbb{P}^5$ by equation (1.170) is a hypercubic defined by the equation

$$\begin{vmatrix} x^{00} & x^{01} & x^{02} \\ x^{10} & x^{11} & x^{12} \\ x^{20} & x^{21} & x^{22} \end{vmatrix} = 0, \quad x^{ij} = x^{ji}, \tag{1.174}$$

and called the *cubic symmetroid*. This hypercubic is a bisecant variety for the Veronese surface $V(2)$. It carries two families of two-dimensional plane generators. One of these families consists of conisecant planes of the surface $V(2)$, and the second consists of two-dimensional planes tangent to this surface. The Veronese surface $V(2)$ is the manifold of singular points of the cubic symmetroid (1.174).

In Section 2.5 we will prove that the Veronese variety and the cubic symmetroid are mutually dual submanifolds.

NOTES

1.2. For more detail on differentiable manifolds, see, for example, the books [KN 63] by Kobayashi and Nomizu or [Di 71] by Dieudonné or [Va 01] by Vasil'ev and on the theory of systems of Pfaffian equations in involution the books [BCGGG 91] by Bryant, Chern, Gardner, Goldsmith, and Griffiths, [C 45] by Cartan, [Fi 48] by Finikov, [Gr 83] by Griffiths, [GJ 87] by Griffiths and Jensen, and [AG 93] by Akivis and Goldberg.

A more detailed presentation of the foundations of the theory of affine connections can be found in the books [KN 63] by Kobayashi and Nomizu and [Lich 55] by Lichnerowicz (see also the papers [Lap 66, 69] by Laptev).

1.3. For more detail on the notion of a multidimensional projective space, see the book [Di 64] by Dieudonné and the paper [GH 79] by Griffiths and Harris.

1.4. The method of moving frames was first used by Frenet [Fr 47] and Serret [Se 51], who applied it to the theory of curves in the Euclidean plane and the Euclidean space. Following this, Darboux [Da] applied this method to the theory of surfaces in the Euclidean space. Cartan developed the method of specialization of moving frames for studying submanifolds in any homogeneous space (see, for example, [C 35] and [C 45]). In Russia, Finikov (see [Fi 48, 50]) and his students widely used the method of specialization of moving frames in their work.

In this book we systematically use the method of moving frames and make specializations of moving frames when they are appropriate.

1.5. On the Grassmann coordinates, see, for example, the book [HP 47] by Hodge and Pedoe.

On Veronese variety, see the book [SR 85] by Semple and Roth. The embedding (1.171) generating the Veronese variety was considered in many papers and books from different points of view (see, for example, the book [GH 78] by Griffiths and Harris and the papers [CDK 70] by Chern, do Carmo, and Kobayashi, [EH 87] by Eisenbud and Harris, [GH 79] by Griffiths and Harris, [J 89] by Jijtchenko, [LP 71] by Little and Pohl, [Nom 76] by Nomizu, [NY 74] by Nomizu and Yano, [Sas 91] by Sasaki, [SegC 21a, 21b, 22] by C. Segre, [Sev 01] by Severi, and recent papers [K 00a, 00b] by Konnov).

The book [Ha 92] by Harris contains an excellent presentation of different properties of the Grassmannians (see Lecture 6), the determinant varieties (see Lecture 12), the Segre varieties (see Lectures 2 and 18), the Veronese varieties (see Lectures 2 and 18), and many other special algebraic varieties.

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Chapter 2

Varieties in Projective Spaces and Their Gauss Maps

In this chapter, after introducing in Sections 2.1 and 2.2 the basic notions (such as the tangent, osculating and normal subspaces, the second fundamental tensor and the second fundamental form, and the asymptotic lines and asymptotic cone) associated with a variety in a projective space $\mathbb{P}^N$, in Section 2.3, we define the rank of a variety and varieties with degenerate Gauss maps. In Section 2.4, we consider the main examples of varieties with degenerate Gauss maps (cones, torses, hypersurfaces, joins, etc.). In Section 2.5, we study the duality principle and its applications, consider another example of varieties with degenerate Gauss maps (the cubic symmetroid) and correlative transformations, and in Section 2.6, we investigate a hypersurface with a degenerate Gauss map associated with a Veronese variety and find its singular points.

2.1 Varieties in a Projective Space

2.1.1 Equations of a Variety. Let M be an n-dimensional connected differentiable manifold, and let f be a nondegenerate almost everywhere differentiable mapping of M into a projective space $\mathbb{P}^N$:

$$f : M \to \mathbb{P}^N,$$

where $n < N$. The image $X = f(M)$ of the manifold M under this mapping is also differentiable almost everywhere. We shall call X an *n-dimensional variety* (or sometimes *subvariety*). Note that the manifold M is differentiable while the variety $X = f(M)$ is almost everywhere differentiable.

For a point $x \in X$ of a variety $X \subset \mathbb{P}^N$, we have $\dim T_x X \geq \dim X = n$. If $\dim T_x X = \dim X = n$, then a point x is called *regular* (or *smooth*), and if $\dim T_x X > \dim X = n$, a point x is called *singular* (see Shafarevich [Sha 88], Chapter 2, §1).

We denote the *locus of smooth points* of X by X_{sm} and the *locus of singular points* of X by $\operatorname{Sing} X$, so

$$X_{sm} = \{x \in X : \dim T_x X = \dim X\},$$
$$\operatorname{Sing} X = \{x \in X : \dim T_x X > \dim X\}.$$

It is obvious that $\operatorname{Sing} X \subset X$, $X_{sm} \subseteq X$, $\dim X_{sm} = n$, dim $\operatorname{Sing} X < n$.

If t^i, $i = 1, \ldots, n$, are differentiable coordinates on the manifold M, then the variety X can be given by the equations

$$x^u = x^u(t^i), \quad u = 0, 1, \ldots, N, \tag{2.1}$$

where $x^u(t^i)$ are almost everywhere differentiable functions of the variables t^i, and the rank of the matrix $(\frac{\partial x^u}{\partial t^i})$ does not exceed n. Because x^u are homogeneous coordinates of a point x of the space $\mathbb{P}^N$, the functions x^u admit multiplication by a common factor, which can be not only a number but also a function $f(t^i)$.

The locus of singular points $\operatorname{Sing} X$ is determined by the condition

$$\operatorname{rank}\left(\frac{\partial x^u}{\partial t^i}\right) < n.$$

The variety X can also be given locally by a system consisting of $N - n$ independent equations of the form

$$F^\alpha(x^0, x^1, \ldots, x^N) = 0, \quad \alpha = n + 1, \ldots, N, \tag{2.2}$$

where F^α are homogeneous almost everywhere differentiable functions. In a neighborhood of a nonsingular point x, the Jacobi matrix $\left(\frac{\partial F^\alpha}{\partial x^u}\right)$ is of rank $N - n$. Hence without loss of generality, we may assume that if in a neighborhood of a point $x \in X$, $\det\left(\frac{\partial F^\alpha}{\partial x^\beta}\right) \neq 0, \alpha, \beta = n + 1, \ldots, N$, then equations (2.2) can be solved for the variables x^α:

$$x^\alpha = x^\alpha(x^0, x^1, \ldots, x^n), \quad \alpha = n + 1, \ldots, N. \tag{2.3}$$

Here the right-hand sides are homogeneous functions of first degree. Therefore, these right-hand sides and the right-hand sides of equations (2.1) contain n essential variables that determine the location of a point on the variety X. If we set $x^i/x^0 = t^i$, we reduce equations (2.3) to the form (2.1).

In Section 1.5 we considered some algebraic submanifolds in a projective space. Certainly, those are differentiable manifolds. Moreover, equations (1.163) defining the image $\Omega(m,n)$ of the Grassmannian $\mathbb{G}(m,n)$ in the space $\mathbb{P}^N$, where $N = \binom{n+1}{m+1} - 1$, are of form (2.2), and equations (1.165) and (1.171), defining the Segre and Veronese varieties, respectively, are of form (2.1). However, the parameters in equations (1.165) and (1.171) are homogeneous while the parameters in equations (2.1) are nonhomogeneous. But as we indicated for equation (2.3), in a neighborhood of a nonsingular point, it is easy to change homogeneous parameters for nonhomogeneous ones.

2.1.2 The Bundle of First-Order Frames Associated with a Variety. Let X be an almost everywhere differentiable variety of dimension n in the projective space $\mathbb{P}^N$, and let x be its nonsingular point. In what follows, we assume that a point $x \in X$ under consideration is nonsingular without also specifying this. Consider all smooth curves passing through a point $x \in X_{sm}$. The tangent lines to these curves at the point x lie in an n-dimensional subspace $T_x(X)$ of the space $\mathbb{P}^N$, called the *tangent subspace to the variety X at the point x*. For brevity, we also use the symbol T_x for the subspace $T_x(X)$.

If x is a regular point of the variety X, then the tangent subspace $T_x(X)$ can be considered in two ways: as a vector space L^{n+1} formed by the vectors $\boldsymbol{v} = \overline{xy}$, where $y \in T_x(X)$ or as a projective subspace $\mathbb{P}^n$ of the projective space $\mathbb{P}^N$ with the fixed point $x \in X$. In what follows, we will adhere to the second point of view. Unless otherwise stated, we will conduct all our considerations in a neighborhood of a regular point $x \in X$.

We associate a family of moving frames $\{A_u\},\ u = 0, 1, \ldots, N$, with each point $x \in X_{sm}$, and assume that for all these frames the point A_0 coincides with the point x, and the points $A_i,\ i = 1, \ldots, n$, lie in the tangent subspace T_x. The frames of this family are called *first-order frames*. Because the differential $dx = dA_0$ of the point x belongs to the tangent subspace T_x, its decomposition with respect to the vertices of the frame $\{A_u\}$ can be written as:

$$dA_0 = \omega_0^0 A_0 + \omega_0^i A_i. \tag{2.4}$$

Thus, in the space $\mathbb{P}^N$, the variety X along with the family of first-order frames is defined by the following system of Pfaffian equations:

$$\omega_0^\alpha = 0,\ \alpha = n+1, \ldots, N, \tag{2.5}$$

and the forms ω_0^i in equation (2.4) are linearly independent and form a cobasis in the tangent subspace T_x. For brevity, we denote these forms by ω^i:

$$\omega_0^i = \omega^i.$$

We call equations (2.5) the *basic* equations of the variety X.

By the structure equations (1.73) of a projective space $\mathbb{P}^N$ and by equations (2.5), the exterior differentials of the forms ω^i can be written as

$$d\omega^i = \omega^j \wedge (\omega_j^i - \delta_j^i \omega_0^0). \tag{2.6}$$

This implies that the 1-forms

$$\theta_j^i = \omega_j^i - \delta_j^i \omega_0^0 \tag{2.7}$$

are the base forms of the frame bundle $\mathcal{R}^1(M)$ of first-order frames on the manifold M of parameters of the variety X. The forms ω^i are the basis forms of the manifold M as well as of the variety X. By relation (1.64), if the point x is held fixed, the forms ω^i satisfy the differential equations

$$\delta\omega^i + \omega^j(\pi_j^i - \delta_j^i \pi_0^0) = 0, \tag{2.8}$$

where, as in Chapter 1, the symbol δ denotes the restriction of the differential d to the fiber $\mathcal{R}_x^1$ of the frame bundle $\mathcal{R}^1(M)$, and $\pi_v^u = \omega_v^u(\delta)$.

If the point x is held fixed on the variety X, then the forms ω^i vanish, $\omega^i = 0$. In this case, the tangent subspace T_x is also fixed. Hence the forms ω_i^α also vanish. Thus, if the point x is held fixed, then the admissible transformations of the moving frames are determined by the following derivational equations:

$$\begin{cases} \delta A_0 = \pi_0^0 A_0, \\ \delta A_i = \pi_i^0 A_0 + \pi_i^j A_j, \\ \delta A_\alpha = \pi_\alpha^0 A_0 + \pi_\alpha^i A_i + \pi_\alpha^\beta A_\beta. \end{cases} \tag{2.9}$$

The 1-forms $\pi_0^0, \pi_i^0, \pi_i^j, \pi_\alpha^0, \pi_\alpha^i$ and π_α^β in (2.9) define the group of transformations of first-order frames associated with the point $x = A_0$. This group is called the *stationary subgroup* of the plane element (x, T_x) of X.

Because the family of first-order frames is associated with each point x of the variety X, the *bundle* $\mathcal{R}^1(X)$ *of first-order frames* is defined on the whole variety X. The base of this bundle is the variety X itself, its base forms are the forms ω^i, its typical fiber is a set of first-order frames associated with a point $x = A_0$, and its fiber forms are the forms $\omega_0^0, \omega_i^0, \omega_i^j, \omega_\alpha^0, \omega_\alpha^i$, and ω_α^β.

Consider the projectivization $\widetilde{T}_x = T_x/A_0$ of the tangent subspace T_x with the center $A_0 = x$ (see Section 1.3.3). This projectivization is a projective space $\widetilde{\mathbb{P}}^{n-1}$ whose elements are the straight lines of the space T_x passing through the point x.

As indicated in Section 1.3, this projectivization defines an equivalence relation in the set of points of the space T_x. This explains why it is natural to denote this projectivization by T_x/A_0:

$$\widetilde{T}_x = T_x/A_0.$$

A frame in the space $\widetilde{T}_x = \widetilde{\mathbb{P}}^{n-1}$ is formed by the points $\widetilde{A}_i = A_i/A_0$, and the forms ω^i become homogeneous coordinates of the point $\widetilde{Y} \in \widetilde{\mathbb{P}}^{n-1}$, i.e.,

$$\widetilde{Y} = \omega^i \widetilde{A}_i.$$

Consider also the projectivization of the space $\mathbb{P}^N$ with the tangent subspace T_x as the center of projectivization. The elements of this projectivization are $(n+1)$-dimensional subspaces of the space $\mathbb{P}^N$ containing the n-dimensional subspace T_x. We denote this projectivization by $\widetilde{\mathbb{P}}^{N-n-1} = \mathbb{P}^N/T_x$. The basis points of the space $\widetilde{\mathbb{P}}^{N-n-1}$ are the points $\widetilde{A}_\alpha = A_\alpha/T_x$, determined by $(n+1)$-dimensional subspaces passing through the points A_α and the center T_x of projectivization. The space $\widetilde{\mathbb{P}}^{N-n-1} = \mathbb{P}^N/T_x$ is called the *first normal subspace* of the variety X at its point x and is denoted by $N_x(X) = \mathbb{P}^N/T_x$.

2.1.3 The Prolongation of Basic Equations. The further investigation of a variety X in a projective space $\mathbb{P}^N$ is concerned with differential prolongations of equations (2.5) defining this variety along with the family of first-order moving frames associated with it. Exterior differentiation of these equations gives the exterior quadratic equations

$$\omega^i \wedge \omega_i^\alpha = 0. \tag{2.10}$$

Applying the Cartan lemma to these exterior equations, we obtain the expressions of the forms ω_i^α in terms of the basis forms ω^i of the variety X:

$$\omega_i^\alpha = b_{ij}^\alpha \omega^j, \;\; b_{ij}^\alpha = b_{ji}^\alpha. \tag{2.11}$$

The 1-forms $\{\omega_0^\alpha, \omega_i^\alpha\}$ are the basis forms of the Grassmannian $\mathbb{G}(n, N)$ whose elements are the subspaces $p = A_0 \wedge A_1 \wedge \ldots \wedge A_n$. But on the variety X, we have $\omega^\alpha = 0$ (see (2.5)). Thus, equation (2.11) defines a mapping of the variety X into the Grassmannian $\mathbb{G}(n, N)$. This mapping is called the *Gauss map.* We denote it by γ:

$$\gamma : X \to \mathbb{G}(n, N).$$

Its name is related to the fact that this map is a projective generalization of the spherical map, introduced by Gauss, of a surface V^2 of a three-dimensional Euclidean space R^3 into a sphere S^2 by means of unit normal vectors.

To establish the nature of the geometric object with the components b^α_{ij}, we evaluate the exterior differentials of equations (2.11) by means of structure equations (1.73) of the space $\mathbb{P}^N$. This results in the following exterior equations:

$$\nabla b^\alpha_{ij} \wedge \omega^j = 0, \tag{2.12}$$

where

$$\nabla b^\alpha_{ij} = db^\alpha_{ij} - b^\alpha_{kj}\theta^k_i - b^\alpha_{ik}\theta^k_j + b^\beta_{ij}\theta^\alpha_\beta, \tag{2.13}$$

and the forms θ^j_i are determined by formulas (2.7). As we noted earlier, these forms are connected with transformations of the first-order frames in the subspace $T_x(M)$ tangent to the manifold M of parameters of the variety X. Similarly, the forms

$$\theta^\alpha_\beta = \omega^\alpha_\beta - \delta^\alpha_\beta\omega^0_0 \tag{2.14}$$

determine admissible transformations of moving frames in the space $N_x(X)$.

Applying the Cartan lemma to exterior quadratic equation (2.12), we obtain the equations

$$\nabla b^\alpha_{ij} = b^\alpha_{ijk}\omega^k, \tag{2.15}$$

where the coefficients b^α_{ijk} are symmetric with respect to all lower indices. It follows from these equations that if $\omega^i = 0$, we have

$$\nabla_\delta b^\alpha_{ij} = \delta b^\alpha_{ij} - b^\alpha_{kj}\sigma^k_i - b^\alpha_{ik}\sigma^k_j + b^\beta_{ij}\sigma^\alpha_\beta = 0, \tag{2.16}$$

where

$$\sigma^j_i = \pi^j_i - \delta^j_i\pi^0_0, \;\; \sigma^\alpha_\beta = \pi^\alpha_\beta - \delta^\alpha_\beta\pi^0_0.$$

Comparing equations (2.16) with equations (1.13), we see that the quantities b^α_{ij} form a tensor relative to the indices i and j. They also form a tensor relative to the index α under transformations of moving frames in the space $N_x(X)$. Such tensors are called *mixed tensors*.

2.2 The Second Fundamental Tensor and the Second Fundamental Form

2.2.1 The Second Fundamental Tensor, the Second Fundamental Form, and the Osculating Subspace of a Variety. The tensor b^α_{ij} is connected with the second-order differential neighborhood of a point x of the variety X. For this reason, this tensor is called the *second fundamental tensor of the variety* X. Let us clarify the geometric meaning of this tensor. To do this, we compute the second differential of the point $x = A_0$ by differentiating the relation (2.4):

$$d^2A_0 = (d\omega_0^0 + (\omega_0^0)^2 + \omega_0^i\omega_i^0)A_0 + (\omega_0^0\omega_0^i + \omega_0^j\omega_j^i)A_i + \omega_0^i\omega_i^\alpha A_\alpha. \qquad (2.17)$$

Factorizing the latter relation by the tangent subspace $T_x = A_0 \wedge A_1 \wedge \ldots \wedge A_n$, we obtain

$$d^2A_0/T_x = \omega_0^i\omega_i^\alpha \widetilde{A}_\alpha, \qquad (2.18)$$

where $\widetilde{A}_\alpha$ are basis points of the normal space $N_x = \mathbb{P}^N/T_x$.

Substituting the values of ω_i^α from equations (2.11) into equation (2.18) and denoting the left-hand side by Φ, we find that

$$\Phi = b_{ij}^\alpha \omega^i\omega^j \widetilde{A}_\alpha. \qquad (2.19)$$

This expression is a quadratic form with respect to the coordinates ω^i, having values in the normal subspace N_x. The form Φ is called the *second fundamental form* of the variety X. Thus, the second fundamental form defines a mapping of the tangent subspace $T_x(X)$ into the normal subspace $N_x(X)$:

$$\Phi : \mathrm{Sym}^{(2)} T_x(X) \to N_x(X).$$

This mapping is called the *Meusnier–Euler mapping* (see Griffiths and Harris [GH 79]).

Note that *a variety X is an n-plane or a part of an n-plane if and only if the second fundamental form Φ vanishes on X*. In fact, if $\Phi \equiv 0$, then it follows from formula (2.18) that $\omega_i^\alpha = 0$ on X. This implies that the equations of infinitesimal displacement of a moving frame become:

$$\begin{cases} dA_0 = \omega_0^0 A_0 + \omega^i A_i, \\ dA_i = \omega_i^0 A_0 + \omega_i^j A_j, \end{cases}$$

and as a result, the n-plane $A_0 \wedge A_1 \wedge \ldots \wedge A_n$ is fixed, and the point A_0 moves in this n-plane.

The scalar forms

$$\Phi^\alpha = b_{ij}^\alpha \omega^i\omega^j \qquad (2.20)$$

are the coordinates of the form Φ with respect to the moving frame $\{\widetilde{A}_\alpha\}$ in the space N_x. Let us denote the maximal number of linearly independent forms Φ^α by m. In some instances, it is convenient to consider the bundle of second fundamental forms of the variety X defined by the relation

$$\Phi(\xi) = \xi_\alpha b_{ij}^\alpha \omega^i\omega^j, \qquad (2.21)$$

where $\xi = (\xi_\alpha)$. The number m is the dimension of this bundle. In what follows, we assume that the number m is constant on the variety X.

The quantities ξ_α occurring in (2.21) define a hyperplane $\xi = \xi_\alpha x^\alpha = 0$, which is tangent to the variety X at the point x, and expression (2.21) is called the *second fundamental form of the variety X with respect to the hyperplane ξ*.

In the space N_x, consider the points

$$\widetilde{B}_{ij} = b^\alpha_{ij} \widetilde{A}_\alpha. \tag{2.22}$$

Because $\widetilde{B}_{ij} = \widetilde{B}_{ji}$, the number of these points is equal to $\frac{1}{2}n(n+1)$. However, it is not necessarily the case that all these points are linearly independent. The maximal number of linearly independent points $\widetilde{B}_{ij}$ coincides with the maximal number of linearly independent forms Φ^α, which we denoted by m. Note that according to our general point of view (see the Preface), we suppose that the integer m is the same on the entire variety X in question, and we will make similar assumptions relative to all other integer-valued invariants arising in our further considerations.

It is obvious that the number m satisfies the following inequalities:

$$0 \leq m \leq \frac{n(n+1)}{2} \quad \text{and} \quad m \leq N - n. \tag{2.23}$$

In the space N_x, the points $\widetilde{B}_{ij}$ span the subspace $\widetilde{\mathbb{P}}^{m-1}$.

Next, in the space $\mathbb{P}^N$, we consider the subspace, which is the linear span of the subspace T_x and the points $B_{ij} = b^\alpha_{ij} A_\alpha$. By relation (2.17), this subspace is also the linear span of all two-dimensional osculating planes of all curves of the variety X passing through the point x. For this reason, this subspace is called the *second osculating subspace* of the variety X at its point x, and it is denoted by $T^{(2)}_x$. We consider the tangent subspace T_x as the *first osculating subspace* of the variety X at a point x, $T_x = T^{(1)}_x$.

2.2.2 Further Specialization of Moving Frames and Reduced Normal Subspaces. We will make a further specialization of moving frames $\{A_u\}$ associated with a point $x \in X$. To do this, we place the vertices $A_{n+1}, \ldots, A_{n+m}$ of the frames into the second osculating subspace $T^{(2)}_x$, whose dimension is equal to $n + m$. The frames thus obtained are called the *frames of second order.*

With this specialization, the points B_{ij}, which together with the points A_0 and A_i define the second osculating subspace $T^{(2)}_x$, are expressed in terms of the points A_{i_1} alone: $B_{ij} = b^{i_1}_{ij} A_{i_1}$, $i_1 = n+1, \ldots, n+m$. So, we have

$$b^{\alpha_1}_{ij} = 0, \quad \alpha_1 = n + m + 1, \ldots, N, \tag{2.24}$$

and therefore formulas (2.11) break up into two groups:

$$\omega_i^{i_1} = b_{ij}^{i_1}\omega^j, \tag{2.25}$$

$$\omega_i^{\alpha_1} = 0. \tag{2.26}$$

Therefore the second fundamental forms Φ^α of the variety X can be written as follows:

$$\Phi^{i_1} = b_{ij}^{i_1}\omega^i\omega^j, \quad \Phi^{\alpha_1} = 0, \tag{2.27}$$

and formula (2.18) becomes

$$d^2 A_0/T_x = \omega^i\omega_i^{i_1}\widetilde{A}_{i_1}. \tag{2.28}$$

The forms Φ^{i_1} are linearly independent, and the matrix $(b_{ij}^{i_1})$ of coefficients of these forms, having m rows and $\frac{1}{2}n(n+1)$ columns, is of rank m.

Consider now the projectivization with the center T_x of the projective space $T_x^{(2)}$. This projectivization is a projective space of dimension $m-1$. We call this space the *reduced first normal subspace* of the variety X and denote it by $\widetilde{N}_x$:

$$\widetilde{N}_x = T_x^{(2)}/T_x. \tag{2.29}$$

If $N > n+m$, then at the point $x \in X$ it is also possible to define the *second normal subspace*

$$N_x^{(2)} = \mathbb{P}^N/T_x^{(2)}, \tag{2.30}$$

whose dimension is equal to $N-n-m-1$ and whose basis is formed by the points $\widetilde{A}_{\alpha_1} = A_{\alpha_1}/T_x^{(2)}$.

Let us now establish the form of equations (2.15) after the specialization of moving frames indicated earlier. These equations also break into two groups:

$$\nabla b_{ij}^{i_1} = db_{ij}^{i_1} - b_{kj}^{i_1}\theta_i^k - b_{ik}^{i_1}\theta_j^k + b_{ij}^{j_1}\theta_{j_1}^{i_1} = b_{ijk}^{i_1}\omega^k, \tag{2.31}$$

$$\nabla b_{ij}^{\alpha_1} = b_{ij}^{i_1}\omega_{i_1}^{\alpha_1} = b_{ijk}^{\alpha_1}\omega^k. \tag{2.32}$$

Equations (2.31) show that the quantities $b_{ij}^{i_1}$ form a tensor relative to the indices i, j, and i_1. Because the matrix $(b_{ij}^{i_1})$ is of rank m, equations (2.32) can be solved with respect to the forms $\omega_{i_1}^{\alpha_1}$:

$$\omega_{i_1}^{\alpha_1} = c_{i_1 k}^{\alpha_1}\omega^k. \tag{2.33}$$

Substituting these expressions of the forms $\omega_{i_1}^{\alpha_1}$ into equations (2.32), we obtain

$$b_{ij}^{i_1} c_{i_1 k}^{\alpha_1} = b_{ijk}^{\alpha_1}. \tag{2.34}$$

Because the quantities $b^{\alpha_1}_{ijk}$ are symmetric with respect to the indices j and k, we find from (2.34) that

$$b^{i_1}_{ij} c^{\alpha_1}_{i_1 k} = b^{i_1}_{ik} c^{\alpha_1}_{i_1 j}. \tag{2.35}$$

This equation can also be obtained as a result of exterior differentiation of equations (2.26).

In the same manner as for the tensor b^{α}_{ij}, we can prove that the quantities $b^{\alpha_1}_{ijk}$ form a tensor relative to the indices i, j, k, and α_1. This and the relations (2.34) imply that the quantities $c^{\alpha_1}_{i_1 k}$ also form a tensor relative to the indices k, i_1, and α_1. As to the quantities $b^{i_1}_{ijk}$ in relations (2.31), it is easy to verify that they do not form a tensor, but rather they depend on the choice of the subspace $A_0 \wedge A_{n+1} \wedge \ldots \wedge A_{n+m}$, which is complementary to the subspace T_x in the osculating subspace $T^{(2)}_x$.

2.2.3 Asymptotic Lines and Asymptotic Cone. A curve on a two-dimensional surface V^2 of a Euclidean space E^3 is called *asymptotic* if its osculating planes coincide with the tangent planes to the surface V^2 or are undetermined (see, for example, Blaschke's books [Bl 21], p. 52, or [Bl 50], p. 65). This definition is projectively invariant and can be generalized to the case where we have a variety of any dimension n in a projective space $\mathbb{P}^N$. Namely, a curve l on a variety X is said to be *asymptotic* if its two-dimensional osculating plane at any of its points x belongs to the tangent subspace T_x to the variety X at this point or is undetermined.

If a curve l is given on the variety X by a parametric equation $x = x(t)$, then its osculating plane is determined by the points $x(t), x'(t)$ and $x''(t)$. But because $x = A_0$, this plane can also be defined by the points A_0, dA_0, and $d^2 A_0$. Because for an asymptotic line the second differential of its point belongs to the tangent subspace T_x, it follows from equation (2.17) that on this curve we have

$$\Phi = \omega^i \omega^{\alpha}_i A_{\alpha} = 0, \tag{2.36}$$

i.e., the second fundamental form of the variety X vanishes on l. Thus in coordinate form, the equations of asymptotic lines have the form

$$b^{\alpha}_{ij} \omega^i \omega^j = 0. \tag{2.37}$$

On a curve l passing through the point x, the basis forms ω^i have the form $\omega^i = \xi^i dt$, where ξ^i are coordinates of a tangent vector to the curve. Substituting these expressions into equations (2.37), we obtain

$$b^{\alpha}_{ij} \xi^i \xi^j = 0. \tag{2.38}$$

These equations define a cone C_x of directions with vertex x. This cone belongs to the tangent subspace T_x and is called the *asymptotic cone*.

If we place the points A_{i_1}, $i_1 = n+1, \ldots, n+m$, of our moving frames into the second osculating subspace $T_x^{(2)}$, as we did in Section 2.2.2, then by (2.38), the equations of the asymptotic cone C_x at the point x can be written as

$$b_{ij}^{i_1} \xi^i \xi^j = 0, \quad i_1 = n+1, \ldots, n+m. \tag{2.39}$$

The problem of existence of asymptotic directions at the point x of the variety X is reduced to finding nontrivial solutions of the system of equations (2.39). This is an algebraic problem. In general, nontrivial solutions exist if $m \leq n-1$. However, in some special cases, nontrivial solutions of equations (2.39) may exist even if $m > n-1$.

2.2.4 The Osculating Subspace, the Second Fundamental Form, and the Asymptotic Cone of the Grassmannian. As an example, we now consider the second osculating subspace and the second fundamental form for the Grassmannian $\mathbb{G}(m,n)$.

As in Section 1.4, we denote by $\Omega(m,n)$ the image of the Grassmannian $\mathbb{G}(m,n)$ under the Grassmann mapping. This image is a variety of dimension $\rho = (m+1)(n-m)$ in the projective space $\mathbb{P}^N$, where $N = \binom{n+1}{m+1} - 1$.

With each element $p = \mathbb{P}^m$ of $\mathbb{G}(m,n)$ we associate a family of moving frames whose points A_i, $i = 0, 1, \ldots, m$, span the subspace $\mathbb{P}^m$. Then we have

$$dA_i = \omega_i^j A_j + \omega_i^\alpha A_\alpha, \quad \alpha = m+1, \ldots, n, \tag{2.40}$$

where ω_i^α are the basis forms of $\mathbb{G}(m,n)$.

The subspace $\mathbb{P}^m$ can be represented as

$$p = A_0 \wedge A_1 \wedge \ldots \wedge A_m, \tag{2.41}$$

where the symbol $\wedge$ denotes the exterior product. Differentiating (2.41) and using (2.40), we obtain

$$dp = \omega p + \omega_i^\alpha p_\alpha^i, \tag{2.42}$$

where $\omega = \omega_0^0 + \omega_1^1 + \ldots + \omega_m^m$, and

$$p_\alpha^i = A_0 \wedge A_1 \wedge \ldots \wedge A_{i-1} \wedge A_\alpha \wedge A_{i+1} \wedge \ldots \wedge A_m.$$

This implies that the tangent subspace T_p to the variety $\Omega(m,n)$ is the span of the points p and p_α^i.

Formula (2.42) proves that the forms ω_i^α are coordinates of a point in the projective space T_p/p with respect to the moving frame $\widetilde{p}_\alpha^i = p_\alpha^i/p$.

To find the second differential of the point p, we first differentiate the points p_α^i and then apply projectivization with the center T_p. This gives

$$dp_\alpha^i / T_p = \omega_j^\beta \widetilde{p}_{\alpha\beta}^{ij}, \tag{2.43}$$

where

$$\widetilde{p}^{ij}_{\alpha\beta} = p^{ij}_{\alpha\beta}/T_p$$

and

$$p^{ij}_{\alpha\beta} = A_0 \wedge A_1 \wedge \ldots \wedge A_{i-1} \wedge A_\alpha \wedge A_{i+1} \wedge \ldots \wedge A_{j-1} \wedge A_\beta \wedge A_{j+1} \wedge \ldots \wedge A_m.$$

Thus, the points $p^{ij}_{\alpha\beta}$ are skew-symmetric with respect to both the upper and lower indices. By equation (2.43), the projectivization with the center T_p of the second differential of the point p has the form

$$d^2 p/T_p = \frac{1}{2} \sum_{\alpha,\beta,i,j} (\omega_i^\alpha \omega_j^\beta - \omega_i^\beta \omega_j^\alpha) \widetilde{p}^{ij}_{\alpha\beta}. \tag{2.44}$$

The right-hand side of this expression is the second fundamental form Φ of the image $\Omega(m,n)$ of the Grassmannian $\mathbb{G}(m,n)$. The coordinates of this form are written as follows:

$$\omega^{\alpha\beta}_{ij} = \omega_i^\alpha \omega_j^\beta - \omega_i^\beta \omega_j^\alpha. \tag{2.45}$$

It follows that the forms $\omega^{\alpha\beta}_{ij}$ are skew-symmetric in both the upper and lower indices. If $i < j$ and $\alpha < \beta$, the points $p^{ij}_{\alpha\beta}$ are linearly independent, and their number is equal to $\rho_1 = \binom{m+1}{2}\binom{n-m}{2}$. The number of linearly independent forms $\omega^{\alpha\beta}_{ij}$ is equal to the same number ρ_1. The points p, p^i_α, and $p^{ij}_{\alpha\beta}$ determine the second osculating subspace $T_p^{(2)}$ of the variety $\Omega(m,n)$ at the point p. Because the dimension of the tangent space T_p of $\Omega(m,n)$ is equal to

$$\dim T_p = (m+1)(n-m) = \binom{m+1}{1}\binom{n-m}{1}, \tag{2.46}$$

the dimension of its second osculating subspace $T_p^{(2)}$ is given by the formula:

$$\dim T_p^{(2)} = \binom{m+1}{1}\binom{n-m}{1} + \binom{m+1}{2}\binom{n-m}{2}. \tag{2.47}$$

The equation of the asymptotic cone C of the variety $\Omega(m,n)$ has the form

$$\omega^{\alpha\beta}_{ij} = \omega_i^\alpha \omega_j^\beta - \omega_i^\beta \omega_j^\alpha = 0. \tag{2.48}$$

Because the forms $\omega^{\alpha\beta}_{ij}$ are the minors of second order of the rectangular matrix

$$M = (\omega_i^\alpha), \tag{2.49}$$

equations (2.48) are equivalent to the conditions

$$\text{rank}\ \ M = 1. \tag{2.50}$$

But as we noted in Section 1.4, in the projective space T_p/p this condition defines the Segre variety $S(m-1, n-m-1)$ carrying plane generators of dimensions $m-1$ and $n-m-1$. The Segre variety $S(m-1, n-m-1)$ is the projectivization of the asymptotic cone C, which is the *Segre cone* $C(m, n-m)$. The vertex of this cone is the point p, and its director manifold is the Segre variety $S(m-1, n-m-1)$.

2.2.5 Varieties with One-Dimensional Normal Subspaces. Consider an n-dimensional variety $X = V^n$ belonging to a projective space $\mathbb{P}^{n+1}$. Such a variety is called a *hypersurface*. For a hypersurface X, equations (2.5), (2.11), and (2.20) have the forms

$$\omega_0^{n+1} = 0, \tag{2.51}$$

$$\omega_i^{n+1} = b_{ij}\omega^j, \ \ b_{ij} = b_{ji}, \tag{2.52}$$

$$\Phi = b_{ij}\omega^i\omega^j, \tag{2.53}$$

where $b_{ij} = b_{ij}^{n+1}$ is the second fundamental tensor of the hypersurface X.

If $\Phi \equiv 0$ at any point $x \in X$, then as we showed in Section 2.2.1, the hypersurface X coincides with its first osculating subspace, i.e., it degenerates into a hyperplane.

If the form Φ does not identically vanish, then the osculating subspace $T_x^{(2)}$ coincides with the space $\mathbb{P}^{n+1}$. Moreover, in this case, the normal subspace N_x is of dimension 1 and coincides with the reduced normal subspace $\widetilde{N}_x$. The hypersurface X has a single relatively invariant second fundamental form Φ, which at any point x determines the cone $C_x \subset T_x$ of asymptotic directions with vertex at x. The cone C_x is defined by the equation

$$\Phi = b_{ij}\omega^i\omega^j = 0. \tag{2.54}$$

Consider a variety $X = V^n$ in the space $\mathbb{P}^N$, and suppose that all second fundamental forms Φ^α, $\alpha = n+1, \ldots, N$, of X are proportional. In this case, the points of the variety X are called *axial*, and the reduced normal subspaces $\widetilde{N}_x$ of X are of dimension 1, as was the case for a hypersurface.

Specializing the moving frames in the same way as in Section 2.2.2, we obtain

$$\Phi^{n+1} = b_{ij}\omega^i\omega^j, \tag{2.55}$$

$$\Phi^{\alpha_1} = 0, \ \alpha_1 = n+2, \ldots, N. \tag{2.56}$$

Thus, equations (2.25) and (2.26) have the form

$$\omega_i^{n+1} = b_{ij}\omega^j, \quad \omega_i^{\alpha_1} = 0. \tag{2.57}$$

Because now the index i_1 takes only one value, formula (2.33) can be written as follows:

$$\omega_{n+1}^{\alpha_1} = c_k^{\alpha_1}\omega^k, \qquad \alpha_1 = n+2, \dots, N, \tag{2.58}$$

and formula (2.52) can be written as

$$b_{ij}c_k^{\alpha_1} = b_{ik}c_j^{\alpha_1}. \tag{2.59}$$

We can now prove the following result.

Theorem 2.1. *If all points of a variety $X = V^n$ of a projective space $\mathbb{P}^N$ are axial, then either the variety X belongs to its fixed osculating subspace $T_x^{(2)}$ of dimension $n+1$, or this variety is a torse, i.e., it is an envelope of a one-parameter family of n-dimensional subspaces.*

Proof. Suppose that rank $\Phi = r \geq 2$. Then the matrix of this form can be reduced to a diagonal form, i.e., $b_{ij} = 0,\ i \neq j, b_{aa} \neq 0, b_{uu} = 0, a = 1, \dots r;$ $u = r+1, \dots, n$. As a result, equations (2.59) take the form

$$b_{aa}c_k^{\alpha_1} = 0, \quad k \neq a.$$

But because the index a takes more than one value, this implies that

$$c_k^{\alpha_1} = 0 \text{ for any } \ k = 1, \dots, n.$$

Thus, we have $\omega_{n+1}^{\alpha_1} = 0$, the subspace $T_x^{(2)} = A_0 \wedge A_1 \wedge \dots \wedge A_n \wedge A_{n+1}$ remains fixed when the point x moves along the variety X, and $X \subset T_x^{(2)}$.

If rank $\Phi = r = 1$, then the matrix of Φ can be reduced to the form in which

$$b_{11} \neq 0,\ b_{ij} = 0 \text{ if } i \neq 1 \text{ or } j \neq 1.$$

As a result, equations (2.59) take the form

$$b_{11}c_k^{\alpha_1} = 0, \quad k \neq 1.$$

It follows that $c_k^{\alpha_1} = 0$, and the forms $\omega_{n+1}^{\alpha_1}$ become

$$\omega_{n+1}^{\alpha_1} = c_1^{\alpha_1}\omega^1.$$

Thus, the family of tangent subspaces T_x of the variety X depends on one parameter, and therefore this variety is a torse (see Example 2.5 in Section 2.4). □

In the case when X is a variety of an N-dimensional space of constant curvature, a similar theorem was proved by C. Segre (see [SegC 07], p. 571), and for this reason, it is called the *Segre theorem.* The proof given above implies that the result of Segre's theorem does not depend on a metric but is of pure projective nature. So our theorem is a *generalized Segre theorem.*

2.3 Rank and Defect of Varieties with Degenerate Gauss Maps

To a regular point $x \in X \subset \mathbb{P}^N$, there corresponds the tangent subspace T_x. Because T_x is an element of the Grassmannian $\mathbb{G}(n, N)$, the variety X defines a map

$$\gamma : \; X \to \mathbb{G}(n, N). \tag{2.60}$$

As we said earlier, under this map, we have $\gamma(x) = T_x(X)$. We called the map γ the *Gauss map.*

We denote the image of the variety X under the Gauss map γ by $\gamma(X)$. Denote by r the rank of the Gauss map $\gamma(X)$, rank $\gamma(X) = r$. It is obvious that $0 \leq r \leq n$. The *rank* of the variety X is defined as the rank of the map γ: rank $X =$ rank $\gamma(X)$.

Because $T_x = A_0 \wedge A_1 \wedge \ldots \wedge A_n$, the basis forms of the Grassmannian $\mathbb{G}(n, N)$ are the forms $\{\omega_0^\alpha, \omega_i^\alpha\}$. Thus, the Gauss map $\gamma(X)$ is defined by equations (2.5) and (2.11). It follows from these equations that

$$\operatorname{rank} \gamma(X) = \operatorname{rank} X = \operatorname{rank} \,(\omega_i^\alpha) = \operatorname{rank} \,(b_{ij}^\alpha \omega^j). \tag{2.61}$$

Let $x \in X$ be a regular point of a variety $X \subset \mathbb{P}^N$, and Φ_x be its second fundamental form at this point. Consider the subspace

$$T'_x = \{\xi \in T_x | \Phi_x(\xi, \eta) = 0 \;\text{ for any }\; \eta \in T_x\}.$$

By (2.20), in a coordinate form, this subspace is defined by the system of equations

$$b_{ij}^\alpha \xi^i = 0. \tag{2.62}$$

The number $l = \dim T'_x$ is called the *Gauss defect* of a variety X (see the book [FP 01], p. 89, by Fischer and Piontkowski) or the *index of relative nullity* of the second fundamental form Φ of the variety X at the point x (see the paper [CK 52] by Chern and Kuiper).

Comparing equations (2.61) and (2.62), we find that

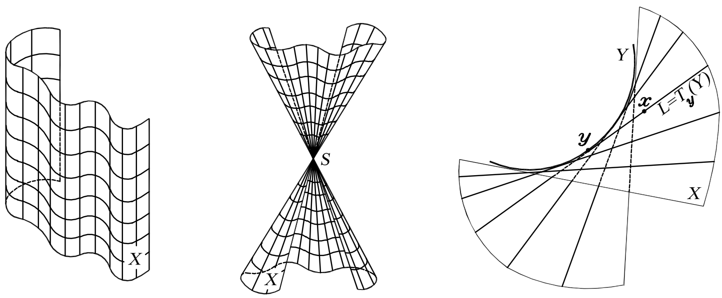
$$l + r = n,$$

i.e., *the sum of the defect and the rank of a variety X coincides with its dimension.*

In what follows, we assume that at all points $x \in X$, its rank (and therefore its defect) takes a constant value.

If $r = \operatorname{rank} X = n$, then the Gauss map γ is nondegenerate. In this case, the tangent subspace $T_x(X)$ to the variety X depends on n parameters, and the variety X is called *tangentially nondegenerate*. For such a variety, the forms ω_i^α in equations (2.11) cannot be expressed in terms of fewer than n linearly independent forms ω^i.

If $r = \operatorname{rank} X < n$, then the Gauss map γ is degenerate. In this case, its Gauss image $\gamma(X)$ depends on r parameters, where $0 \leq r < n$. Then we say that the variety X is *tangentially degenerate of rank* r, or X is a *variety with a degenerate Gauss map of rank* r. We denote such variety by $X = V_r^n$, $\operatorname{rank} X = r < n$. Varieties with a degenerate Gauss map of rank r foliate into their leaves L of dimension $l = n - r$, along which the tangent subspace $T_x(X)$ is fixed. This foliation is called the Monge–Ampère foliation (see Section 3.1.1). We will prove in Theorem 3.1 (see Section 3.1.3) that the leaves of this foliation are l-planes.



(a) (b) (c)

Figure 2.1

In a three-dimensional Euclidean space $\mathbb{E}^3$ $(N = 3, n = 2, r = 1)$ varieties with degenerate Gauss maps are known as *developable surfaces*. There are three classes of developable surfaces in E^3: cylinders, cones, and tangent developables of space curves (see Figure 2.1 $(a), (b), (c)$).

If $\operatorname{rank} X = 0$, then the matrix (b_{ij}^α) is the zero matrix, the form Φ is also 0, $\Phi = 0$, and a variety X is a flat variety, i.e., X *is an n-dimensional projective subspace $\mathbb{P}^n$ of the space $\mathbb{P}^N$, or it is an open part of $\mathbb{P}^n$.*

2.4 Examples of Varieties with Degenerate Gauss Maps

Consider a few examples of varieties with degenerate Gauss maps.

Example 2.2. If $\operatorname{rank} X = \dim X = n$, then X is a variety of complete rank. X is also called *tangentially nondegenerate* in the space $\mathbb{P}^N$. Such varieties do not have singular points.

For example, the quadric Q defined in a three-dimensional projective space $\mathbb{P}^3$ by the equation

$$x_0 x_3 - x_1 x_2 = 0$$

is tangentially nondegenerate. For the quadric Q, we have $n = 2, N = 3$, $r = 2, l = 0$. Such a quadric bears two families of rectilinear generators. However, the tangent plane $T(Q)$ is not constant along these generators, i.e., none of these families compose the Monge–Ampère foliation.

Example 2.3. As we showed in Section 2.2.1, for $r = 0$, a variety X is an n-dimensional subspace $\mathbb{P}^n$, $n < N$. This variety is the only variety with a degenerate Gauss map without singularities in $\mathbb{P}^N$.

Example 2.4. Suppose that S is a subspace of the space $\mathbb{P}^N$, $\dim S = l - 1$, and T is its complementary subspace, $\dim T = N - l$, $T \cap S = \emptyset$. Let Y be a smooth tangentially nondegenerate variety of the subspace T, $\dim Y = \operatorname{rank} Y = r < N - l$. Consider an r-parameter family of l-dimensional subspaces $L_y = S \wedge y$, $y \in Y$. This variety is a *cone* X with vertex S and the director manifold Y. The subspace $T_x(X)$ tangent to the cone X at a point

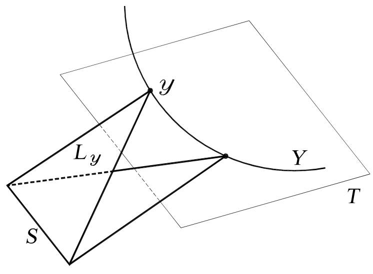


Figure 2.2

$x \in L_y (x \notin S)$ is defined by its vertex S and the subspace $T_y(Y)$, $T_x(X) = S \wedge T_y(Y)$, and $T_x(X)$ remains fixed when a point x moves in the subspace L_y. As a result, the cone X is a variety with a degenerate Gauss map of dimension $n = l + r$ and rank r, with plane generators L_y of dimension l (see

Figure 2.2). The generators L_y of the cone X are leaves of the Monge–Ampère foliation associated with X.

Example 2.5. Consider a smooth curve Y in the space $\mathbb{P}^N$ not belonging to a subspace $\mathbb{P}^{l+1} \subset \mathbb{P}^N$ and the set of its osculating subspaces L_y of order and dimension l. This set forms a variety $X = \cup_{y \in Y} L_y$ of dimension $l+1$ and rank $r = 1$ in $\mathbb{P}^N$. Such a variety is called a *torse* (cf. Section 2.2.5). The subspace $T_y = L_y + \frac{dL_y}{dy}$ is the tangent subspace to X at all points of its generator L_y. Thus, the subspaces L_y are the leaves of the Monge–Ampère foliation associated with the torse X. The subspace $F_y = L_y \cap \frac{dL_y}{dy}$ describes also a torse of dimension l. This process of construction of torses departing from X can be continued in both directions: from one side until we reach a smooth curve Y for which the subspace L_y is the osculating subspace of order $l-1$, and from the other side until we reach an $(N-1)$-dimensional variety (hypersurface) with a degenerate Gauss map. Figure 2.3 shows a torse in $\mathbb{P}^3$.

Conversely, a variety of dimension n and rank 1 is a torse formed by a family of osculating subspaces of order $n-1$ of a curve of class C^p, $p \geq n-1$, in the space $\mathbb{P}^N$.

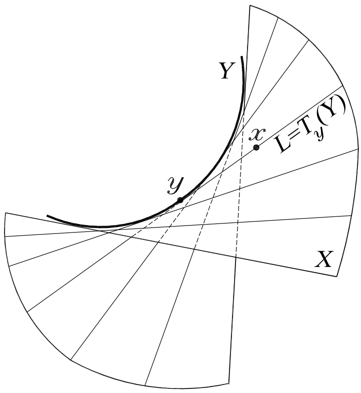


Figure 2.3

In what follows, unless otherwise stated, we always assume that $r > 1$.

In particular, we consider the spatial third-degree curve[1] C defined in the space $\mathbb{P}^3$ by the parametric equations $x(t) = (t^3, t^2, t, 1)$. The tangent line to C is determined by the point $x(t)$ and the point $x'(t) = (3t^2, 2t, 1, 0)$. The parametric equations of this tangent line have the form

$$y(t,s) = x(t) + sx'(t) = (t^3 + 3t^2 s, t^2 + 2ts, t+s, 1).$$

[1]Cayley [Cay 64] called such a curve a *twisted cubic*.

A surface swept by these tangent lines is a torse—a variety with a degenerate Gauss map of rank one and dimension two in the space $\mathbb{P}^3$. In this case we have $n = 2$, $N = 3$, $l = 1$, $r = 1$. The tangents to the line $x(t)$ are the leaves of the Monge–Ampère foliation associated with this third-degree curve.

In order to obtain an equation of form (2.2) of the torse X formed by the tangents to the third-degree curve, we need to exclude parameters t and s from the parametric equations of the third-degree curve and its tangent line. An equation of this torse can also be obtained by a method indicated by Cayley (see [Cay 64]).

Let (y_0, y_1, y_2, y_3) be homogeneous coordinates of the space $\mathbb{P}^3$. Consider the nonhomogeneous polynomial

$$\psi(t) := y_0 t^3 + y_1 t^2 + y_2 t + y_3.$$

An osculating plane of the third-degree curve $x(t) = (t^3, t^2, t, 1)$ is defined by the points $x(t), x'(t) = (3t^2, 2t, 1, 0)$, and $x''(t) = (6t, 2, 0, 0)$. So, the equation of this plane is

$$\begin{vmatrix} y_0 & y_1 & y_2 & y_3 \\ t^3 & t^2 & t & 1 \\ 3t^2 & 2t & 1 & 0 \\ 6t & 2 & 0 & 0 \end{vmatrix} = 0$$

or

$$\psi^*(t) := y_0 - 3y_1 t + 3y_2 t^2 - y_3 t^3 = 0.$$

It follows from this form of $\psi^*(t)$ that the dual curve $x^*(t)$ has the parameterization $x^*(t) = (1, -3t, 3t^2, -t^3)$. Its osculating plane is defined by the points $x^*(t), (x^*)'(t) = (0, -3, 6t, -3t^2)$, and $(x^*)''(t) = (0, 0, 6, -6t)$. Thus, its equation is

$$\begin{vmatrix} y_0 & y_1 & y_2 & y_3 \\ 1 & -3t & 3t^2 & -t^3 \\ 0 & -3 & 6t & -3t^2 \\ 0 & 0 & 6 & -6t \end{vmatrix} = 0.$$

Easy computation shows that this equation is

$$\psi(t) = 0.$$

The torse X is the envelope of the family of osculating planes $\psi^*(t) = 0$ of the third-degree curve $x(t)$, and the torse X^* is the envelope of the family of osculating planes $\psi(t) = 0$ of the dual curve $x^*(t)$.

We find equations of both torses X and X^*.

According to Cayley [Cay 64], an equation of the torse X^* is

$$\operatorname{Disct} \psi(t) = 0,$$

where $\operatorname{Disct} \psi(t)$ is the discriminant of the polynomial $\psi(t)$. Computing the discriminant

$$\operatorname{Disct} \psi(t) = \begin{vmatrix} y_0 & y_1 & y_2 & y_3 & 0 \\ 0 & y_0 & y_1 & y_2 & y_3 \\ 3y_0 & 2y_1 & y_2 & 0 & 0 \\ 0 & 3y_0 & 2y_1 & y_2 & 0 \\ 0 & 0 & 3y_0 & 2y_1 & y_2 \end{vmatrix}$$

up to a factor of y_0, we obtain the following equation of the torse X^*:

$$\Psi := 27y_0^2y_3^2 - 18y_0y_1y_2y_3 + 4y_0y_2^3 + 4y_1^3y_3 - y_1^2y_2^2 = 0.$$

We can find an equation of the torse X (which is the envelope of the osculating planes $\psi^*(t) = 0$ of the third-degree curve $x^*(t)$) by computing the discriminant of the polynomial $\psi^*(t)$.

However, it is easier to find this equation by making the substitution

$$y_0 \to y_3, \quad y_1 \to -3y_2, \quad y_2 \to 3y_1, \quad y_3 \to -y_0$$

in the equation of the torse X. The result is

$$\Psi^* := y_0^2y_3^2 - 6y_0y_1y_2y_3 + 4y_0y_2^3 + 4y_1^3y_3 - 3y_1^2y_2^2 = 0.$$

This equation shows that the surface swept by the tangents to the third-degree curve is an algebraic fourth-degree surface.

Note that Cayley [Cay 64] took equations of the family of osculating planes of the torse X in the form

$$y_0t^3 + 3y_1t^2 + 3y_2t^2 + y_3 = 0.$$

Comparing this with $\psi^*(t) = 0$, we see that Cayley used the following parametrization of a third-degree curve: $(1, -t, t^2, -t^3)$. It is easy to check that the equations of the torses X and X^* for Cayley's parameterization are precisely the same as for our parameterization. Namely, the torses X and X^* for Cayley's parameterization are defined by the equations $\Psi^* = 0$ (see [Cay 64]) and $\Psi = 0$, respectively.

In his paper [Ca 64], Cayley found equations of torses formed by the tangents to two special fourth-degree curves (quartics) $u(t)$ and $v(t)$ in the space

$\mathbb{P}^3$. He did not indicate the equations of these fourth-degree curves—he found equations of the torses as envelopes of the families of osculating planes of the dual curves $u^*(t)$ and $v^*(t)$.

In addition, in his paper [Cay 64], Cayley considered in $\mathbb{P}^3$ the fourth-degree curves $u(t) = (81, -27t, 9t^2, t^4)$ and $v(t) = (-2, t, -t^3, 2t^4)$ and found equations of the torses formed by the tangents to these curves. These torses are defined by the algebraic equations

$$y_0^3 y_3^2 + 6y_0^2 y_2^2 y_3 - 24y_0 y_1^2 y_2 y_3 + 9y_0 y_2^4 + 16y_1^4 y_3 - 8y_1^2 y_2^3 = 0$$

and

$$y_0^3 y_3^3 - 12y_0^2 y_1 y_2 y_3^2 - 27y_0^2 y_2^4 - 6y_0 y_1^2 y_2^2 y_3 - 27y_1^4 y_3^2 - 64y_1^3 y_2^3 = 0.$$

These equations can be derived in a way similar to what we used to find the equation $\Psi^*(t) = 0$ of the torse formed by the tangents to the third-degree curve $x(t) = (t^3, t^2, t, 1)$.

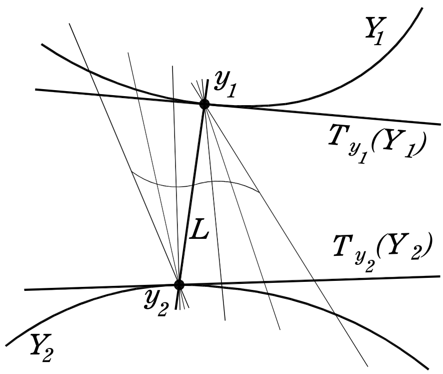


Figure 2.4

Example 2.6. In the space $\mathbb{P}^N$, $N \geq 4$, we take two arbitrary smooth space curves, Y_1 and Y_2, that do not belong to the same three-dimensional space, and the set of all straight lines intersecting these two curves (see Figure 2.4). These straight lines form a three-dimensional variety X. Such a variety is called the *join*. Its dimension is three, dim $X = 3$. It is easy to see that the variety X has a degenerate Gauss map. In fact, the three-dimensional tangent subspace $T_x(X)$ to X at a point x lying on a rectilinear generator L is defined by this generator L and two straight lines tangent to the curves Y_1 and Y_2 at the points y_1 and y_2 of their intersection with the line L. Because this tangent subspace does not depend on the location of the point x on the generator L, the variety under consideration is a variety $X = V_2^3$ with a degenerate Gauss map of rank two.

This example can be generalized by taking k spatial curves in the space $\mathbb{P}^N$, where $N \geq 2k$ and $k > 2$, and considering a k-parameter family of $(k-1)$-planes intersecting all these k curves.

Example 2.7. Let $N = n + 1$, and let Y be an r-parameter family of hyperplanes ξ in general position in $\mathbb{P}^{n+1}$, $r < n$. Such a family has an n-dimensional envelope X that is a variety with a degenerate Gauss map of dimension n and rank r in the space $\mathbb{P}^{n+1}$. It foliates into an r-parameter family of plane generators L of dimension $l = n - r$, along which the tangent subspace $T_x(X)$, $x \in L$, is fixed and coincides with a hyperplane ξ of the family in question. Thus, X is a hypersurface with a degenerate Gauss map of rank r with $(n-r)$-dimensional plane generators L in the space $\mathbb{P}^{n+1}$.

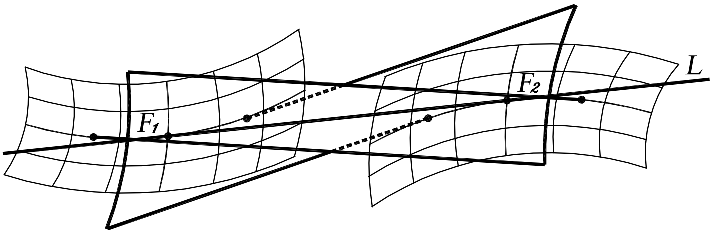


Figure 2.5

Figure 2.5 represents the case $n = 3, r = 2$, i.e., a hypersurface $X = V_2^3 \subset \mathbb{P}^4$.

2.5 Application of the Duality Principle

2.5.1 Dual Variety. For construction of new examples of varieties with degenerate Gauss maps we employ the duality principle in a projective space introduced in Section 1.3.2.

By the duality principle, to a point x of a projective space $\mathbb{P}^N$, there corresponds a hyperplane ξ. A set of hyperplanes of space $\mathbb{P}^N$ forms the dual projective space $(\mathbb{P}^N)^*$ of the same dimension N. Under this correspondence, to a subspace $P \subset \mathbb{P}^N$ of dimension p, there corresponds a subspace $\mathbb{P}^* \subset (\mathbb{P}^N)^*$ of dimension $N - p - 1$. Under the dual map, the incidence of subspaces is reversed, that is, if $\mathbb{P}_1 \subset \mathbb{P}_2$, then $\mathbb{P}_1^* \supset \mathbb{P}_2^*$.

Let X be an irreducible, almost everywhere smooth variety of dimension n, $\dim X = n$, in the space $\mathbb{P}^N$, let x be a smooth point of X, and let $T_x X$ be the tangent subspace to X at the point x. A hyperplane ξ is said to be *tangent to* X *at* x if $T_x \subset \xi$. The bundle of hyperplanes ξ tangent to X at x is of dimension $N - n - 1$.

The set of all hyperplanes ξ tangent to the variety X at its smooth points composes a variety

$$X^\wedge = \{\xi \subset P^N | \exists\, x \in X_{sm} \text{ such that } T_x X \subseteq \xi\}.$$

But this variety can be not closed if X has singular points. The *dual variety* X^* of a variety X is the closure of the variety $X^\wedge$:

$$X^* = \overline{X^\wedge} = \overline{\{\xi \subset P^N | \exists\, x \in X_{sm} \text{ such that } T_x X \subseteq \xi\}}. \tag{2.63}$$

The dual variety X^* can also be described as the envelope of the family of hyperplanes ξ dual to the points $x \in X$. This gives a practical way for finding X^*, which we will use in examples.

If a variety X is tangentially nondegenerate, i.e., if its rank $r = n$, then in the general case, the dimension n^* of its dual variety X^* is equal to

$$n^* = \dim X^* = (N - n - 1) + n = N - 1. \tag{2.64}$$

Equation (2.64) means that the variety X^* is a hypersurface with a degenerate Gauss map in the space $(\mathbb{P}^N)^*$. The rank r of X^* equals the dimension n of the variety X, $r = \operatorname{rank} X^* = n$, and its Gauss defect $\delta_\gamma(X^*) = l^* = n^* - r = N - r - 1$.

However, it may happen that $\dim X^* < N - 1$. Then the number

$$\delta_* = N - 1 - \dim X^*$$

is called the *dual defect* of the variety X, and the variety X itself is said to be *dually degenerate.*

An example of a dually degenerate smooth variety is the Segre variety $X = \text{Seg}\ (\mathbb{P}^m \times \mathbb{P}^n) \subset \mathbb{P}^{mn+m+n}$, whose dual defect equals $|m - n|$ (see Example 2.11).

If a variety X has a degenerate Gauss map (i.e., if its rank $r < n$), then the dual variety X^* is a fibration whose fiber is the bundle $\Xi = \{\xi \subset \mathbb{P}^N | \xi \supseteq T_L X\}$ of hyperplanes ξ containing the tangent subspace $T_L X$ and whose base is the manifold $B = X^*/\Xi$. The dimension of a fiber Ξ of this fibration (as in the case $r = n$) equals $N - n - 1$, $\dim \Xi = N - n - 1$, and the dimension of the base B equals r, $\dim B = r$, i.e., the dimension of B coincides with the rank of the variety X. Therefore, in the general case, the dimension n^* of its dual variety X^* is determined by the formula

$$\dim X^* = (N - n - 1) + r = N - l - 1, \tag{2.65}$$

where $l = \dim L = \delta_\gamma(X) = n - r$, and its Gauss defect is equal to $\delta_\gamma(X^*) = l^* = n^* - r = (N - l - 1) - r = N - n - 1 = \dim \Xi$.

However, it may happen that $\dim X^* < N - l - 1$. Then the number

$$\delta_* = N - l - 1 - \dim X^*$$

is called the *dual defect* of the variety X, and the variety X itself is said to be *dually degenerate*. Note that the dual defect of tangentially nondegenerate varieties (see p. 71) can be obtained from this new definition by taking $l = 0$.

Note also that dually degenerate smooth varieties in the projective space $\mathbb{P}^N$ are few and far between. As to dually degenerate varieties with degenerate Gauss maps, we are aware of only a few examples of dually degenerate varieties X with degenerate Gauss maps: the varieties X with degenerate Gauss maps of ranks three and four in $\mathbb{P}^N$ were considered by Piontkowski [Pio 02b].

This is why *in this book we consider only dually nondegenerate varieties in the space* $\mathbb{P}^N$, i.e., we assume that for the variety $X \subset \mathbb{P}^N$ of dimension n and rank r, the dimension of its dual variety is determined by formula (2.65).

2.5.2 The Main Theorem. The following theorem follows immediately from the preceding considerations.

Theorem 2.8. *Let X be a dually nondegenerate variety with a degenerate Gauss map of dimension n and rank r in the space $\mathbb{P}^N$. Then the leaves L of the Monge–Ampère foliation of X are of dimension $l = n - r$. The dual variety $X^* \subset (\mathbb{P}^N)^*$ is of dimension*

$$n^* = N - l - 1 \tag{2.66}$$

and the same rank r, and the leaves L^* of the Monge–Ampère foliation of X^* are of dimension

$$l^* = N - n - 1. \tag{2.67}$$

Under this map, the plane generator L^* corresponds to a tangent subspace $T_x(X)$ of the variety X, and the tangent subspace $T_\xi(X^*)$ of the variety X^* corresponds to a plane generator L, i.e., on X the tangent bundle $T(X)$ and the Monge–Ampère foliation $L(X)$ are mutually dual.

In particular, if a variety $X \subset \mathbb{P}^N$ is tangentially nondegenerate, then we have $n = r$, $l = 0$ (i.e., $n^* = N - 1$), and the dual map (*) sends X to a hypersurface $X^* \subset (\mathbb{P}^N)^*$ with a degenerate Gauss map of rank n with the leaves L^* of the Monge–Ampère foliation of dimension $l^* = N - n - 1$.

Conversely, if X is a hypersurface with a degenerate Gauss map of rank $r < N-1$ in $\mathbb{P}^N$, then the variety X^* dual to X is a tangentially nondegenerate variety of dimension r and rank r.

In particular, the dual map (*) sends a tangentially nondegenerate variety $X \subset \mathbb{P}^N$ of dimension and rank $r = n = N-2$ to a hypersurface $X^* \subset (\mathbb{P}^{n+2})^*$ with a degenerate Gauss map of rank r, and X^* bears an r-parameter family of rectilinear generators. Each of these rectilinear generators possesses r foci if each is counted as many times as its multiplicity. The hypersurface X^* is torsal and foliates into r families of torses. The original variety X bears a net of conjugate lines corresponding to the torses of the variety X^*. Of course, the correspondence indicated above is mutual.

We consider an irreducible, almost everywhere smooth variety X of dimension n and rank r in the space $\mathbb{P}^N$ in more detail. The tangent bundle $T(X)$ of X is formed by the n-dimensional subspaces T_x tangent to X at points $x \in X$ and depending on r parameters. The subspaces T_x are tangent to X along the plane generators L of dimension $l = n - r$ composing on X the Monge–Ampère foliation $L(X)$. The bundle $T(X)$ and the foliation $L(X)$ have a common r-dimensional base.

Let (*) be the dual map of $\mathbb{P}^N$ onto $(\mathbb{P}^N)^*$. The dual map (*) sends the variety X to a variety X^*, which is the set of all hyperplanes $\xi \subset (\mathbb{P}^N)^*$ tangent to X along the leaves L of its Monge–Ampère foliation. The map (*) sends the tangent bundle $T(X)$ and the Monge–Ampère foliation $L(X)$ of X to the Monge–Ampère foliation $L(X^*)$ and the tangent bundle $T(X^*)$ of X^*, respectively. Thus, under the dual map (*), we have

$$(T(X))^* = L(X^*), \quad (L(X))^* = T(X^*),$$

where $\dim T(X^*) = \dim X^* = n^* = N - l - 1$ and $\dim L(X^*) = \dim L^* = l^* = N - n - 1$.

We now consider a few examples.

Example 2.9. First, we consider a simple example. Let X be a smooth spatial curve X in a three-dimensional projective space $\mathbb{P}^3$. For this curve, we have $N = 3$, $n = r = 1$, $l = 0$, and $T_x(X)$ is the tangent line to X at x. The dual map (*) sends a point $x \in X$ to a plane $\xi \subset X^*$, and the dual variety X^* is the envelope of the one-parameter family of hyperplanes ξ (see Figure 2.6), i.e, X^* is a torse.

Using the formulas for n^* and l^* written earlier we find that $n^* = 2$, $l^* = 1$. The variety X^* bears rectilinear generators L^* along which the tangent planes

$\xi = T(X^*)$ are constant. Hence rank $X^* = 1$. The generators L^* of the torse X^* are dual to the tangent lines $T(X)$ to the curve X.

Next, we determine which varieties correspond to the varieties with degenerate Gauss maps considered in Examples 2.4, 2.5, and 2.7.

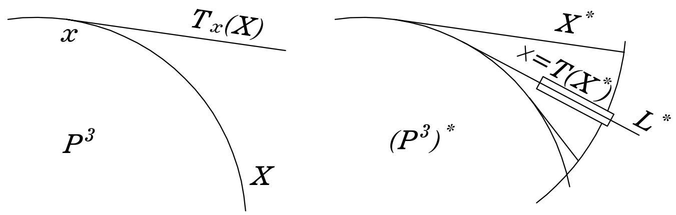


Figure 2.6

Example 2.10. To a cone X of rank r with vertex S of dimension $l-1$ (see Example 2.4), there corresponds a variety X^* lying in the subspace $T = S^*$, $\dim T = N - l$. Because $\dim X^* = n^* = N - l - 1$, the variety X^* is a hypersurface of rank r in the subspace T. Such a hypersurface was considered in Example 2.7.

If a tangentially nondegenerate variety X, $\dim X = \operatorname{rank} X = r$, belongs to a subspace $\mathbb{P}^{n+1} \subset \mathbb{P}^N$, then we can consider two dual maps in the spaces $\mathbb{P}^{n+1}$ and $\mathbb{P}^N$. We denote the first of these maps by $*$ and the second by $\circ$. Then under the first map, the image of X is a hypersurface $X^* \subset \mathbb{P}^{n+1}$, and under the second map, the hypersurface X is transferred into a cone X° of rank r and dimension $n^\circ = N - n + r - 1$ with an $(N-n-2)$-dimensional vertex $S = (\mathbb{P}^{n+1})^\circ$ and $(N-n-1)$-dimensional plane generators $L^\circ = T(X)^\circ$. It follows that Examples 2.4 and 2.7 are mutually dual to each other.

For the torse X (see Example 2.5), we have $n = l+1$, $r = 1$ and $n^* = N - l - 1$, $l^* = N - l - 2$, i.e., the dual image X^* of a torse X is a torse.

Thus, the varieties considered in Examples 2.4, 2.5, and 2.7 are dual to varieties considered in 2.7, 2.5, and 2.4, respectively.

Example 2.11. *The Segre variety* (see Griffiths and Harris [GH 79] and Tevelev [T 01]) $S(m,n)$ is the embedding of the direct product of the projective spaces $\mathbb{P}^m$ and $\mathbb{P}^n$ in the space $\mathbb{P}^{mn+m+n}$:

$$S : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n},$$

defined by the equations

$$z^{ik} = x^i y^k,$$

where $i = 0, 1, \ldots, m$, $k = 0, 1, \ldots, n$, and x^i, y^k, and z^{ik} are the coordinates of points in the spaces $\mathbb{P}^m, \mathbb{P}^n$, and $\mathbb{P}^{mn+m+n}$, respectively. This manifold has the dimension $m+n$, $\dim S(m,n) = m+n$.

Consider in the spaces $\mathbb{P}^m$ and $\mathbb{P}^n$ projective frames $\{A_0, A_1, \ldots, A_m\}$ and $\{B_0, B_1, \ldots, B_n\}$. Then in the space $\mathbb{P}^{mn+m+n}$ we obtain the projective frame

$$\{A_0 \otimes B_0,\, A_0 \otimes B_k,\, A_i \otimes B_0,\, A_i \otimes B_k\}$$

(here and in what follows $i, j = 1, \ldots, m$; $k, l = 1, \ldots, n$) consisting of $(m+1)(n+1)$ linearly independent points of the space $\mathbb{P}^{mn+m+n}$. The point $A_0 \otimes B_0$ is the generic point of the variety S.

In the spaces $\mathbb{P}^m$ and $\mathbb{P}^n$, we have the following equations:

$$dA_0 = \omega_0^0 A_0 + \omega_0^i A_i, \;\; dB_0 = \sigma_0^0 B_0 + \sigma_0^k B_k$$

(see (1.71)). Hence

$$d(A_0 \otimes B_0) = (\omega_0^0 + \sigma_0^0)(A_0 \otimes B_0) + \omega_0^i (A_i \otimes B_0) + \sigma_0^k (A_0 \otimes B_k),$$

and the subspace in $\mathbb{P}^{mn+m+n}$ spanned by the points $A_0 \otimes B_0, A_i \otimes B_0$, and $A_0 \otimes B_k$ is the tangent subspace to the Segre variety S at the point $A_0 \otimes B_0$:

$$T_{A_0 \otimes B_0} = \text{Span}\,(A_0 \otimes B_0,\, A_i \otimes B_0,\, A_0 \otimes B_k).$$

The second differential of the point $A_0 \otimes B_0$ has the form:

$$d^2(A_0 \otimes B_0) = 2\,\omega_0^i\,\sigma_0^k\,A_i \otimes B_k \quad (\text{mod}\; T_{A_0 \otimes B_0}).$$

Hence the osculating subspace $T^2_{A_0 \otimes B_0}(S)$ to the variety S coincides with the entire space $\mathbb{P}^{mn+m+n}/T_{A_0 \otimes B_0}$, and its second fundamental forms have the form

$$\Phi^{ik} = \omega_0^i \sigma_0^k.$$

The total number of these forms is mn. The equations $\omega_0^i = 0$ determine n-dimensional plane generators on S, and the equations $\sigma_0^k = 0$ determine its m-dimensional plane generators.

Consider a tangent hyperplane to the Segre variety S at the point $A_0 \otimes B_0$. Because such a hyperplane contains the tangent subspace $T_{A_0 \otimes B_0}$, its equation can be written in the form

$$\xi = \xi_{ik} z^{ik} = 0,$$

where $i = 1, \ldots, m$; $k = 1, \ldots, n$, and z^{ik} are coordinates of points in the space $\mathbb{P}^{mn+m+n}/T_{A_0 \otimes B_0}$. As a result, the second fundamental form of the variety S with respect to the hyperplane ξ is

$$\Phi(\xi) = \xi_{ik}\,\omega_0^i\,\sigma_0^k$$

(see (2.21)). The forms $\Phi(\xi)$ constitute the system of the second fundamental forms of the variety S. The mn forms Φ^{ik} are linearly independent forms of this system. The matrix of this system of second fundamental forms has the form

$$\Xi = \begin{pmatrix} 0 & (\xi_{ik}) \\ (\xi_{ki}) & 0 \end{pmatrix}.$$

In this formula (ξ_{ik}) is a rectangular $(m \times n)$-matrix and (ξ_{ki}) is its transpose.

It follows that $\det \Xi = 0$ if $m \neq n$. In this case, the system of the second fundamental forms of the variety S is degenerate, and the dual defect $\delta_*(S)$ of S equals $|n - m| : \delta_*(S) = |n - m|$. The variety S is dually nondegenerate if and only if $m = n$.

2.5.3 Cubic Symmetroid. Now we consider the Veronese variety given as the image of the embedding

$$V^* : \text{Sym}\,(\mathbb{P}^{2*} \times \mathbb{P}^{2*}) \to \mathbb{P}^{5*}$$

into the projective space $\mathbb{P}^{5*}$. This embedding is defined by the equations

$$x_{ij} = u_i u_j, \quad i, j = 0, 1, 2, \tag{2.68}$$

where u_i are projective coordinates in the plane $\mathbb{P}^{2*}$, i.e., tangential coordinates in the plane $\mathbb{P}^2$, and x_{ij} are projective coordinates in the space $\mathbb{P}^{5*}$, $x_{ij} = x_{ji}$.

Let us find an equation of the variety V that is dual to the variety $V^* \subset \mathbb{P}^{5*}$ defined by equations (2.68). This variety V is the envelope of the two-parameter family of hyperplanes defined in the space $\mathbb{P}^5$ by the equation

$$\xi = x^{ij} u_i u_j = 0, \quad i, j = 0, 1, 2. \tag{2.69}$$

Equation (2.69) depends on two affine parameters $u = \frac{u_1}{u_0}$ and $v = \frac{u_2}{u_0}$, and the quantities x^{ij} occurring in (2.69) are projective coordinates in the space $\mathbb{P}^5$. In order to find the equation of the envelope of the family (2.69), we differentiate equation (2.69) with respect to u_i. The result is

$$\frac{\partial \xi}{\partial u_i} = x^{ij} u_j = 0. \tag{2.70}$$

Eliminating the parameters u_j from equations (2.70), we arrive at the equation

$$\det\,(x^{ij}) = 0,$$

or in more detail,

$$F = \det \begin{pmatrix} x^{00} & x^{01} & x^{02} \\ x^{10} & x^{11} & x^{12} \\ x^{20} & x^{21} & x^{22} \end{pmatrix} = 0. \tag{2.71}$$

Equation (2.71) defines in the space $\mathbb{P}^5$ the cubic hypersurface dual to the Veronese variety (2.68) and called the *cubic symmetroid*.

The Veronese variety V^* defined by equation (2.68) is a tangentially nondegenerate variety in the space $\mathbb{P}^{5*}$. Thus, by Theorem 2.8, its dual variety V is a hypersurface with a degenerate Gauss map of rank two in the space $\mathbb{P}^5$ having two-dimensional leaves $L(V)$ of the Monge–Ampère foliation on V. The latter is dual to the tangent bundle $T(V^*)$ of V^*.

Next we find equations of the leaves $L(V)$ of the cubic symmetroid V. Three hyperplanes

$$\alpha_0 x^{0i} + \alpha_1 x^{1i} + \alpha_2 x^{2i} = 0, \quad i = 0, 1, 2, \tag{2.72}$$

of the space $\mathbb{P}^5$ have a common two-dimensional plane. It is easy to see that the coordinates of points of this 2-plane satisfy equation (2.71). In fact, by (2.72), the rows of the determinant on the left-hand side of (2.71) are linearly dependent, and hence the determinant vanishes. Hence equations (2.72) determine two-dimensional plane generators of the symmetroid V. Because equations (2.72) contain two variables $\frac{\alpha_1}{\alpha_0}$ and $\frac{\alpha_2}{\alpha_0}$, the symmetroid V carries a two-parameter family of two-dimensional plane generators.

The equation of the tangent hyperplane ξ at the point $x = (x^{ij})$ to the cubic symmetroid V defined by equations (2.71) has the form

$$\frac{\partial F}{\partial x^{ij}}\, y^{ij} = 0, \tag{2.73}$$

where y^{ij} are coordinates of an arbitrary point $y \in \xi$.

Equation (2.73) can be written in the form

$$F = x^{00}x^{11}x^{22} + 2x^{01}x^{12}x^{20} - x^{00}(x^{12})^2 - x^{11}(x^{02})^2 - x^{22}(x^{01})^2 = 0. \tag{2.74}$$

By (2.74), the coefficients of equation (2.73) are determined by the formulas

$$\frac{\partial F}{\partial x^{00}} = \begin{vmatrix} x^{11} & x^{12} \\ x^{21} & x^{22} \end{vmatrix}, \quad \frac{\partial F}{\partial x^{01}} = 2\begin{vmatrix} x^{12} & x^{10} \\ x^{22} & x^{20} \end{vmatrix},$$
$$\frac{\partial F}{\partial x^{11}} = \begin{vmatrix} x^{00} & x^{02} \\ x^{20} & x^{22} \end{vmatrix}, \quad \frac{\partial F}{\partial x^{02}} = 2\begin{vmatrix} x^{10} & x^{11} \\ x^{20} & x^{21} \end{vmatrix}, \tag{2.75}$$
$$\frac{\partial F}{\partial x^{22}} = \begin{vmatrix} x^{00} & x^{01} \\ x^{10} & x^{11} \end{vmatrix}, \quad \frac{\partial F}{\partial x^{12}} = 2\begin{vmatrix} x^{01} & x^{00} \\ x^{21} & x^{20} \end{vmatrix}.$$

Consider the plane generators L^0 of the cubic symmetroid V defined by equations (2.72) with $\alpha_0 = \alpha_1 = 0$, $\alpha_2 \neq 0$. For this generator, equations (2.72) take the form

$$x^{2i} = 0. \tag{2.76}$$

This implies that only one coefficient of equation (2.73), namely $\frac{\partial F}{\partial x^{22}}$, is nonvanishing. Hence, equation (2.73) takes the form

$$y^{22} = 0. \tag{2.77}$$

Equation (2.77) is the equation of the tangent hyperplane to V for all points of the generators L^0. As a result, the tangent hyperplane ξ is constant for all points of the generators L^0.

But all plane generators L of the cubic symmetroid V are projectively equivalent. Thus each of them is a leaf of the Monge–Ampère foliation on V, and the symmetroid V itself is a hypersurface with a degenerate Gauss map of rank $r = 2$ in the space $\mathbb{P}^5$. This corresponds to the contents of Theorem 2.8.

2.5.4 Singular Points of the Cubic Symmetroid. Next we find singular points of the cubic symmetroid V defined by equation (2.71). Such points are determined by the equations

$$\frac{\partial F}{\partial x^{ij}} = 0. \tag{2.78}$$

Because all plane generators of the symmetroid V are projectively equivalent, we will look for singular points on the plane generator L^0 defined by equations

(2.76). On this plane generator, all the determinants (2.75) are identically equal to zero, except the determinant $\frac{\partial F}{\partial x^{22}}$. As a result, singular points on the plane generator (2.76) are determined by the equation

$$\frac{\partial F}{\partial x^{22}} = x^{00}x^{11} - (x^{01})^2 = 0. \tag{2.79}$$

Equation (2.79) defines the locus of singular points in the plane generator L^0. Hence, *the locus of singular points in the plane generator* L^0 *is a conic.* Similarly, in all other generators L of the cubic symmetroid V, the loci of singular points are the second-degree curves (the focus curves F_L (see Section 3.2, p. 100) of these generators).

From (2.75) and (2.78) it follows that the set of all singular points on the entire cubic symmetroid V is determined by the equation

$$\operatorname{rank} x^{ij} = 1$$

or

$$x^{ij} = x^i x^j, \quad i, j = 0, 1, 2 \tag{2.80}$$

(cf. equations (2.68)). This means that *the set of singular points of the cubic symmetroid* $V \subset \mathbb{P}^5$ *is a Veronese surface* $V^* \subset \mathbb{P}^{5*}$.

Most likely, all these results are well known in algebraic geometry. However, we obtained them here by the methods of differential geometry.

Now we give one more interpretation of the properties of the cubic symmetroid $V \subset \mathbb{P}^5$. To this end, we denote the entries of the matrix on the left-hand side of (2.71) by a_{ij}, i.e., we write this matrix in the form

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}, \text{ where } a_{ij} = a_{ji}.$$

Because the matrix A is defined up to a nonvanishing factor, in the projective plane $\mathbb{P}^2$, it determines a second-degree curve

$$a_{ij}x^i x^j = 0, \quad i, j = 0, 1, 2$$

(see Figure 2.7 (a)). To the cubic symmetroid V defined in $\mathbb{P}^5$ by the equation

$$\det A = 0, \tag{2.81}$$

there corresponds in $\mathbb{P}^2$ the set of second-degree curves that decompose into two straight lines

$$a_i x^i = 0, \quad b_i x^i = 0, \quad i = 0, 1, 2, \tag{2.82}$$

(see Figure 2.7 (b)).

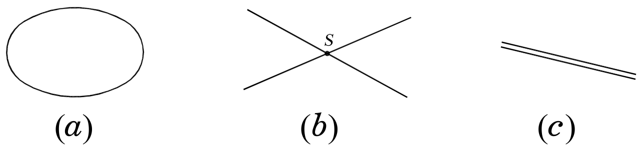


Figure 2.7

To the plane generator $L \subset V$, there corresponds in $\mathbb{P}^2$ the set of second-degree curves of type (2.82) decomposed into two intersecting straight lines with a common point of intersection for all pairs. The family of these plane generators L depends on two parameters because the points of $\mathbb{P}^2$ depend on two parameters.

To a tangent hyperplane of the cubic symmetroid $V \subset \mathbb{P}^5$ at the point $a_{ij} = a_{(i}\, b_{j)}$, there corresponds in $\mathbb{P}^2$ the set of second-degree curves passing through the common point of the straight lines (2.82).

To the set of singular points of the symmetroid $V \subset \mathbb{P}^5$ defined by the equation

$$\operatorname{rank} A = 1,$$

there corresponds in $\mathbb{P}^2$ the set of second-degree curves degenerating into two coinciding straight lines (see Figure 2.7 (c)).

2.5.5 Correlative Transformations. If we have the identification $(\mathbb{P}^N)^* = \mathbb{P}^N$, the duality principle can be realized by a correlative transformation of the space $\mathbb{P}^N$.

Consider a *correlative transformation* $\mathcal{C}$ (a *correlation*) in the space $\mathbb{P}^N$ that maps a point $x \in \mathbb{P}^N$ into a hyperplane $\xi \in \mathbb{P}^N$, $\xi = \mathcal{C}(x)$, and preserves the incidence of points and hyperplanes. A correlation $\mathcal{C}$ maps a k-dimensional subspace $\mathbb{P}^k \subset \mathbb{P}^N$ into an $(N-k-1)$-dimensional subspace $\mathbb{P}^{N-k-1} \subset \mathbb{P}^N$.

We assume that the correlation $\mathcal{C}$ is nondegenerate, i.e., it defines a one-to-one correspondence between points and hyperplanes of the space $\mathbb{P}^N$.

Analytically, a correlation $\mathcal{C}$ can be written in the form

$$\xi_i = c_{ij} x^j, \quad i, j = 0, 1, \dots, N,$$

where x^i are point coordinates and ξ_i are tangential coordinates in the space $\mathbb{P}^N$ (cf. formulas (1.76) on p. 23). A correlation $\mathcal{C}$ is nondegenerate if $\det(c_{ij}) \neq 0$.

Consider a smooth curve C in the space $\mathbb{P}^N$ and suppose that this curve does not belong to a hyperplane. A correlation $\mathcal{C}$ maps points of C into hyperplanes forming a one-parameter family. The hyperplanes of this family envelope a hypersurface with a degenerate Gauss map of rank one with $(N-2)$-dimensional generators (see Figure 2.6 on p. 58).

If the curve C lies in a subspace $\mathbb{P}^s \subset \mathbb{P}^N$, then a correlation $\mathcal{C}$ maps points of C into hyperplanes that envelop a hypercone with an $(N-s-1)$-dimensional vertex.

Further, let $X = V^r$ be an arbitrary tangentially nondegenerate r-dimensional variety in the space $\mathbb{P}^N$. A correlation $\mathcal{C}$ maps points of such V^r into hyperplanes forming an r-parameter family. The hyperplanes of this family envelop a hypersurface $Y = V_r^{N-1}$ with a degenerate Gauss map of rank r. The generators of this hypersurface X are of dimension $N-r-1$ and correspond to the tangent subspaces $T_x(V^r)$.

If the tangentially nondegenerate variety V^r belongs to a subspace $\mathbb{P}^s \subset \mathbb{P}^N$, $s > r$, then the hypersurface $Y = V_r^{N-1}$ corresponding to V^r under a correlation $\mathcal{C}$ is a hypercone with an $(N-s-1)$-dimensional vertex.

Now let $X = V_r^n$ be a variety with a degenerate Gauss map of rank r. Then we can prove the following result, which fully corresponds to Theorem 2.8.

Theorem 2.12. *A correlation $\mathcal{C}$ maps an n-dimensional dually nondegenerate variety $X = V_r^n$ with a degenerate Gauss map of rank r with plane generators of dimension $l = n - r$ into a variety $X^* = V_r^{N-l-1}$, with a degenerate Gauss map of the same rank r with $(N-n-1)$-dimensional plane generators.*

Proof. A correlation $\mathcal{C}$ sends an l-dimensional plane generator $L \subset X$ to an $(N-l-1)$-dimensional plane $\mathbb{P}^{N-l-1}$, and a tangent subspace $T_x(X)$ to an $(N-n-1)$-dimensional plane $\mathbb{P}^{N-n-1}$, where $\mathbb{P}^{N-n-1} \subset \mathbb{P}^{N-l-1}$. Because both of these planes depend on r parameters, the planes $\mathbb{P}^{N-n-1}$ are generators of the variety $\mathcal{C}(X)$, and the planes $\mathbb{P}^{N-l-1}$ are its tangent subspaces. Thus, the variety $\mathcal{C}(X)$ is a variety $X^* = V_r^{N-l-1}$ of dimension $N-l-1$ and rank r. □

2.6 Hypersurface with a Degenerate Gauss Map Associated with a Veronese Variety

2.6.1 Veronese Varieties and Varieties with Degenerate Gauss Maps. Consider a real five-dimensional projective space $\mathbb{RP}^5$ with points whose coordinates are defined by symmetric matrices

$$x = \begin{pmatrix} x^{00} & x^{01} & x^{02} \\ x^{10} & x^{11} & x^{12} \\ x^{20} & x^{21} & x^{22} \end{pmatrix},$$

and its dual space $(\mathbb{RP}^5)^*$ with points whose coordinates are defined by the matrices

$$\xi = \begin{pmatrix} x_{00} & x_{01} & x_{02} \\ x_{10} & x_{11} & x_{12} \\ x_{20} & x_{21} & x_{22} \end{pmatrix} = (x_{ij}),$$

where $i, j = 0, 1, 2$; $x_{ij} = x_{ji}$. In the space $(\mathbb{RP}^5)^*$, a frame consists of the points

$$A^{00} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^{11} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A^{01} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A^{02} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A^{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.83}$$

and an arbitrary point $\xi \in (\mathbb{RP}^5)^*$ can be represented as a linear combination of the vertices of this frame:

$$\xi = x_{ij} A^{ij}.$$

A Veronese variety V in the space $(\mathbb{RP}^5)^*$ can be given by the following parametric equations:

$$\xi = \begin{pmatrix} u^2 & uv & uw \\ vu & v^2 & vw \\ wu & wv & w^2 \end{pmatrix}, \tag{2.84}$$

where (u, v, w) are projective coordinates in the plane $\mathbb{RP}^2$. Thus, the variety V is the embedding

$$\psi : \mathrm{Sym}\,(\mathbb{P}^{2*} \times \mathbb{P}^{2*}) \to \mathbb{P}^{5*}.$$

By (2.83), formula (2.84) can also be written in the form

$$\xi = u^2 A^{00} + v^2 A^{11} + w^2 A^{22} + 2uvA^{01} + 2vwA^{12} + 2uwA^{02}. \tag{2.85}$$

Consider now the projection Pr of the space $(\mathbb{RP}^5)^*$ from the point

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

not belonging to the Veronese variety V, onto the subspace $(\mathbb{RP}^4)^*$ not tangent to the variety V and defined in $(\mathbb{RP}^5)^*$ by the equation

$$x_{00} + x_{11} + x_{22} = 0. \tag{2.86}$$

First, we find the projections of the vertices A^{ij} of the frame of the space $(\mathbb{RP}^5)^*$ onto the subspace $(\mathbb{RP}^4)^*$. Because the vertices A^{01}, A^{12}, and A^{02} belong to the subspace $(\mathbb{RP}^4)^*$, the projections coincide with these points:

$$\operatorname{Pr} A^{01} = A^{01}, \;\; \operatorname{Pr} A^{12} = A^{12}; \;\; \operatorname{Pr} A^{02} = A^{02}$$

(see Figure 2.8).

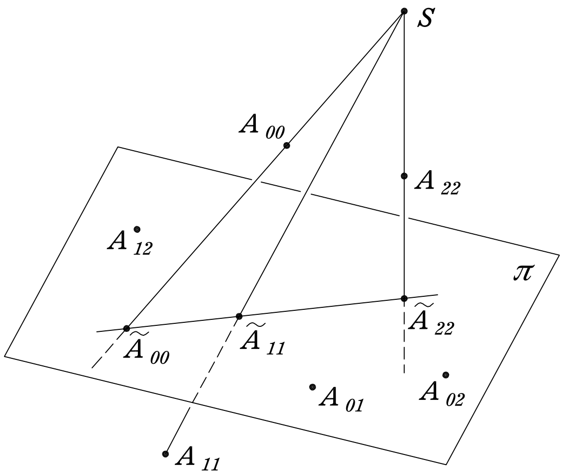


Figure 2.8

The projection of the vertex A_{00} can be found from the condition

$$\operatorname{Pr} A^{00} = A^{00} - \lambda S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in (\mathbb{RP}^4)^*.$$

By (2.86), it follows that

$$1 - 3\lambda = 0, \;\; \lambda = \frac{1}{3},$$

i.e.,

$$\operatorname{Pr} A^{00} = \frac{2}{3} A^{00} - \frac{1}{3} A^{11} - \frac{1}{3} A^{22} = \frac{1}{3} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In a similar way, we find that

$$\operatorname{Pr} A^{11} = \frac{1}{3}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } \operatorname{Pr} A^{22} = \frac{1}{3}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The points $\operatorname{Pr} A^{00}$, $\operatorname{Pr} A^{11}$, and $\operatorname{Pr} A^{22}$ are linearly dependent because

$$\operatorname{Pr} A^{00} + \operatorname{Pr} A^{11} + \operatorname{Pr} A^{22} = 0. \tag{2.87}$$

Thus, we can take the independent points

$$A^{01}, A^{12}, A^{20} \text{ and } \operatorname{Pr} A^{00} = \widetilde{A}^{00}, \operatorname{Pr} A^{22} = \widetilde{A}^{22} \tag{2.88}$$

as a basis of the subspace $(\mathbb{RP}^4)^*$. By (2.87), for the point $\operatorname{Pr} A^{11}$ we obtain the expression

$$\operatorname{Pr} A^{11} = -\widetilde{A}^{00} - \widetilde{A}^{22}. \tag{2.89}$$

Next, we find the projection of the Veronese variety V onto the subspace $(\mathbb{RP}^4)^*$ from the point S. By (2.85), (2.88), and (2.89), we have

$$\operatorname{Pr} \xi = (u^2 - v^2)\widetilde{A}^{00} + (w^2 - v^2)\widetilde{A}^{22} + 2uvA^{01} + 2vwA^{12} + 2uwA^{02}.$$

Note that a similar projection of a Veronese variety into a four-dimensional projective space was considered earlier by Sasaki [Sas 91]. In the space $\mathbb{RP}^4$ dual to the subspace $(\mathbb{RP}^4)^*$, the last equation defines a two-parameter family of hyperplanes ξ corresponding to the points x^* of the space $(\mathbb{RP}^4)^*$. The equation of a hyperplane ξ has the form

$$\xi := (u^2 - v^2)x^{00} + (w^2 - v^2)x^{22} + 2uvx^{01} + 2vwx^{12} + 2uwx^{02} = 0, \tag{2.90}$$

where $x^{00}, x^{22}, x^{01}, x^{12}$, and x^{02} are projective coordinates in the space $\mathbb{RP}^4$. The family of hyperplanes ξ depends on two parameters $\frac{u}{w}$ and $\frac{v}{w}$. Hence the envelope of this family is a hypersurface X with a degenerate Gauss map of rank two in the space $\mathbb{RP}^4$. The hypersurface X bears a two-parameter family of rectilinear generators L that are leaves of the Monge–Ampère foliation on X.

In order to find an equation of the envelope of the family of hyperplanes ξ, we differentiate equation (2.90) with respect to the parameters u, v, and w:

$$\begin{cases} \dfrac{1}{2}\dfrac{\partial \xi}{\partial u} = ux^{00} + vx^{01} + wx^{02} = 0, \\ \dfrac{1}{2}\dfrac{\partial \xi}{\partial v} = ux^{01} - v(x^{00} + x^{22}) + wx^{12} = 0, \\ \dfrac{1}{2}\dfrac{\partial \xi}{\partial w} = ux^{02} + vx^{12} + wx^{22} = 0. \end{cases} \tag{2.91}$$

Because by Euler's theorem on homogeneous functions, we have

$$u\frac{\partial\xi}{\partial u} + v\frac{\partial\xi}{\partial v} + w\frac{\partial\xi}{\partial w} = 2\xi;$$

it follows that by (2.91) equation (2.90) is identically satisfied.

Eliminating the parameters u, v, and w from equations (2.91), we find that

$$\Phi = \det\begin{pmatrix} x^{00} & x^{01} & x^{02} \\ x^{01} & -(x^{00}+x^{22}) & x^{12} \\ x^{02} & x^{12} & x^{22} \end{pmatrix} = 0. \tag{2.92}$$

This equation determines the hypersurface X—the envelope of the family of hyperplanes ξ—in the space $\mathbb{RP}^4$.

This implies the following theorem.

Theorem 2.13. *The hypersurface X dual to the projection of a Veronese variety into a four-dimensional subspace is a cubic hypersurface. This hypersurface has a degenerate Gauss map and is of rank two. It bears a two-parameter family of rectilinear generators that are leaves of the Monge–Ampère foliation on X.*

Moreover, equation (2.92) proves that the hypersurface X is equivalent to the projectivization of the set of symmetric matrices of third order with vanishing determinant and trace.

2.6.2 Singular Points. Let us find singular points of the hypersurface X defined by equation (2.92). In order to do this, we write this equation in the form

$$\begin{aligned} &\Phi = -x^{00}x^{22}(x^{00}+x^{22}) + 2x^{01}x^{02}x^{12} \\ &+(x^{02})^2(x^{00}+x^{22}) - x^{00}(x^{12})^2 - x^{22}(x^{01})^2 = 0. \end{aligned} \tag{2.93}$$

Singular points of the hypersurface X are defined by the equations

$$\frac{\partial\Phi}{\partial x^{00}} = -2x^{00}x^{22} - (x^{22})^2 + (x^{02})^2 - (x^{12})^2 = 0, \tag{2.94}$$

$$\frac{\partial\Phi}{\partial x^{01}} = 2x^{02}x^{12} - 2x^{22}x^{01} = 0, \tag{2.95}$$

$$\frac{\partial\Phi}{\partial x^{12}} = 2x^{01}x^{02} - 2x^{00}x^{12} = 0, \tag{2.96}$$

$$\frac{\partial\Phi}{\partial x^{02}} = 2x^{01}x^{12} + 2x^{02}(x^{00}+x^{22}) = 0, \tag{2.97}$$

and

$$\frac{\partial\Phi}{\partial x^{22}} = -(x^{00})^2 - 2x^{00}x^{22} + (x^{02})^2 - (x^{01})^2 = 0. \tag{2.98}$$

Equations (2.95) and (2.96) imply that

$$x^{12} = \lambda x^{01}, \;\; x^{02} = \lambda x^{00}, \;\; x^{22} = \lambda^2 x^{00}, \tag{2.99}$$

where, of course, $\lambda \neq 0$. Substituting these expressions into equations (2.94), (2.97), and (2.98) and dividing by λ or λ^2, we arrive at the same equation

$$(x^{01})^2 + (1+\lambda^2)(x^{00})^2 = 0,$$

from which it follows that

$$x^{01} = \pm i\sqrt{1+\lambda^2}\, x^{00}. \tag{2.100}$$

Equations (2.99) and (2.100) determine the desired singular points F and $\overline{F}$ on the hypersurface X. These points are complex conjugate on the straight line $F \wedge \overline{F}$.

It is easy to see that the straight line $F \wedge \overline{F}$ belongs to the hypersurface X defined by equation (2.84). In fact, it follows from (2.99) and (2.100) that the coordinates $(x^{00}, x^{22}, x^{01}, x^{12}, x^{02})$ of an arbitrary point $F + s\overline{F}$ of this line are

$$(1+s,\, \lambda^2(1+s),\, i\sqrt{1+\lambda^2}(1-s),\, i\lambda\sqrt{1+\lambda^2}(1-s),\, \lambda(1+s))x^{00}.$$

Substituting these coordinates into the left-hand side of equation (2.93), we obtain zero.

NOTES

2.1–2.2. Our presentation of the projectivization of the tangent and osculating subspaces of a submanifold X is close to that in the paper [GH 79] by Griffiths and Harris (see also the book [AG 93] by Akivis and Goldberg).

The differential geometry of the Grassmannian was considered by Akivis in [A 82].

The osculating spaces, fundamental forms, and asymptotic directions and lines of a submanifold X were investigated by É. Cartan in [C 19]. See more on the second fundamental forms of X in Griffiths and Harris [GH 79] and Landsberg [L 94].

Note that the proof of our Theorem 2.1 is different from that of Theorem 2.2 in [AG 93], which has some inaccuracies.

This theorem generalizes a similar theorem of C. Segre (see [SegC 07], p. 571), which was proved for submanifolds X of dimension n of the space $\mathbb{P}^N$ that have at each point $x \in X$ the osculating subspace T_x^2 of dimension $n+1$. By this theorem, a submanifold X either belongs to a subspace $\mathbb{P}^{n+1}$ or is a torse.

Note that C. Segre proved the theorem named after him for a submanifold of a multidimensional space of constant curvature.

Note also that Theorem 2.1 is similar to Theorem 3.10 from the book [AG 93] by Akivis and Goldberg, which was proved there for submanifolds of a space $\mathbb{P}^N$ bearing a net of conjugate lines.

2.3. Zak [Za 87] (see also his book [Za 93] and the paper [Ra 84] by Ran) proved that the Gauss map of a smooth variety is finite (see also the books [FP 01] by Fischer and Piontkowski (subsections 2.3.3 and 3.1.3); [Ha 92] by Harris (p. 189); [L 99] by Landsberg (p. 48); [T 01] by Tevelev (Sections 3.3 and 4.2); and the book [Za 93] by Zak). In terms of differential geometry, Zak's theorem can be formulated as follows: The image of the Gauss map $\gamma(X)$ of a smooth irreducible variety $X \subset \mathbb{P}^N$ of dimension n, which is different from a linear space, is a smooth irreducible variety $\gamma(X) \subset \mathbb{G}(n, N)$ of the same dimension n.

From the point of view of differential geometry, this result is more or less obvious: If a variety X is smooth in $\mathbb{P}^N$, then its Gauss map $\gamma(X)$ has the rank $r = n$ (i.e., X is tangentially nondegenerate).

Fischer [F 88] extends to the complex analytic case a classical result on ruled surfaces in $\mathbb{E}^3$. He shows that the only developable surfaces in $\mathbb{CP}^3$ are planes, cones, and tangent surfaces of curves. He also shows that a developable ruled surface is uniquely determined by its directrix and its Gauss map.

The origins of the theory of varieties with degenerate Gauss maps are in the works of C. Segre [SegC 07, 10] who studied the local differential geometry of linear spaces. In particular, in [SegC 07, 10], he introduced the Segre cone of such families and used the concepts of the second fundamental forms and foci.

Varieties $X = V_r^n$ with degenerate Gauss maps of rank $r < n$ were considered by É. Cartan in [C 16] in connection with his study of metric deformation of hypersurfaces, and in [C 19] in connection with his study of manifolds of constant curvature; by Yanenko in [Ya 53] in connection with his study of metric deformation of submanifolds of arbitrary class; by Akivis in [A 57, 62], Savelyev in [Sa 57, 60], and Ryzhkov in [Ry 60] (see also the survey paper by Akivis and Ryzhkov [AR 64]) in a projective space $\mathbb{P}^N$. Brauner [Br 38], Wu [Wu 95], and Fischer and Wu [FW 95] studied such varieties in a Euclidean N-space $\mathbb{E}^N$.

Note that a relationship of the rank of varieties V^m and their deformation in a Euclidean N-space was indicated by Bianchi as far back as 1905. In [Bi 05] he proved that a necessary condition for V^m to be deformable is the condition rank $V^m \leq 2$. Allendörfer [Al 39] introduced the notion of type t, $t = 0, 1, , \dots, m$, of V^m and proved that varieties V_{N-p}, $p > 1$, of type $t > 2$ in $\mathbb{E}^N$ are rigid. For definition of type of V^m, see [Al 39] or Yanenko [Ya 53]. Note only that the notion of type (as well as of rank) is projectively and metrically invariant, and that for a hypersurface, the type coincides with the rank.

Griffiths and Harris in [GH 79] (Section 2, pp. 383–393) considered varieties $X = V_r^n$ with degenerate Gauss maps from the point of view of algebraic geometry. Following [GH 79], Landsberg [L 96] considered varieties with degenerate Gauss maps. His recently published book [L 99] is in some sense an update to the paper [GH 79].

Section 5 (pp. 47–50) of these notes is devoted to varieties with degenerate Gauss maps. In the recently published book [FP 01] by Fischer and Piontkowski, the authors studied ruled varieties from the point of view of complex projective algebraic geometry. One section of this book was devoted to varieties with degenerate Gauss maps (they called such varieties developable). Following Griffiths and Harris's paper [GH 79], the authors employed a bilinear second fundamental form for studying developable varieties, gave detailed and more elementary proofs of some results in [GH 79], and reported on some recent progress in this area. In particular, they gave a classification of developable varieties of rank two in codimension one. Rogora in [Rog 97] and Mezzetti and Tommasi in [MT 02a] also considered varieties with degenerate Gauss maps from the point of view of algebraic geometry.

Recently Ishikawa published four papers [I 98, 99a, 99b] and [IM 01] on varieties with degenerate Gauss maps (called "developable" in these papers). In [IM 01], Ishikawa and Morimoto found the connection between such varieties and solutions of Monge–Ampère equations; they named the foliation of plane generators L of X ($\dim L = l$) the Monge–Ampère foliation. In [IM 01], the authors proved that the rank r of a compact C^∞-hypersurface $X \subset \mathbf{R}\,\mathbb{P}^N$ with a degenerate Gauss map is an even integer r satisfying the inequality $\frac{r(r+3)}{2} > N$, $r \neq 0$. In particular, if $r < 2$, then X is necessarily a projective hyperplane of $\mathbf{R}\,\mathbb{P}^N$, and if $N = 3$ or $N = 5$, then a compact C^∞-hypersurface with a degenerate Gauss map is a projective hyperplane.

In [I 98, 99b], Ishikawa found a real algebraic cubic nonsingular hypersurface with a degenerate Gauss map in $\mathbf{R}\mathbb{P}^N$ for $N = 4, 7, 13, 25$, and in [I 99a] he studied singularities of C^∞-hypersurfaces with degenerate Gauss maps.

The notion of the index l of relative nullity was introduced by Chern and Kuiper in their joint paper [CK 52] (see also the book by Kobayashi and Nomizu [KN 63], vol. 2, p. 348) for a variety $X = V^n$ embedded into a Riemannian manifold V^N.

However, the second fundamental forms of a submanifold X are related not so much to the metric structure of X as to its projective structure, because these forms are preserved under projective transformations of the Riemannian submanifold X. This was noticed by Akivis in [A 87b], who also proved the relation $l + r = n$.

Note that if $l > 0$, then the point x is called a *parabolic* point of the variety X. If all points of a variety X are parabolic, then the variety X is called *parabolic* (cf. the papers [Bor 82, 85] by Borisenko). The varieties X, for which the index l is constant and greater than 0 for all points $x \in X$, are called *strongly parabolic.*

In 1997 Borisenko published the survey paper [Bor 97] in which he discussed results on strongly parabolic varieties and related questions in Riemannian and pseudo-Riemannian spaces of constant curvature and, in particular, in a Euclidean space $\mathbb{E}^N$. Among other results, he gives a description of certain classes of varieties of arbitrary codimension that are analogous to the class of parabolic surfaces in a Euclidean space $\mathbb{E}^3$. Borisenko also investigates the local and global metric and topological properties, indicates conditions that imply that a variety of a Euclidean space $\mathbb{E}^N$ is cylindrical, presents results on strongly parabolic varieties in pseudo-Riemannian spaces of constant curvature, and finds the relationship with minimal surfaces.

2.4. The results presented in this section are due to Akivis [A 57] (see also

Section 4.2 in the book [AG 93] by Akivis and Goldberg). In our presentation, we follow the recently published paper [AG 01a] by Akivis and Goldberg. Other examples of varieties with degenerate Gauss maps can be found in the papers [A 87a] by Akivis, [AG 93, 98b, 98c, 01a, 01b, 02b] by Akivis and Goldberg, [AGL 01] by Akivis, Goldberg, and Landsberg, [C 39] by Cartan, [FW 95] by Fischer and Wu, [GH 79] by Griffiths and Harris, [I 98, 99a, 99b, 00a] by Ishikawa, [Pio 01, 02a, 02b] by Piontkowski, [S 60] by Sacksteder, [Wu 95] by Wu, [WZ 02] by Wu and F. Zheng, and in the books [L 99] by Landsberg and [FP 01] by Fischer and Piontkowski. Examples of varieties with degenerate Gauss maps on the sphere S^n were constructed in the recent papers [IKM 01, 02] by Ishikawa, Kimura, and Miyaoka.

2.5. The reader can find more details on the dual varieties and the dual defect of a tangentially nondegenerate variety, for example, in the following books: Fischer and Piontkowski [FP 01] (Sections 2.1.4, 2.1.5, 2.3.4, 2.5.1, 2.5.3, and 2.5.7); Harris [Ha 92] (pp. 196–199); Landsberg [L 99] (pp. 16–17 and 52–57); and Tevelev [T 01] (Chapters 1, 6, and 7). Formula (2.65) for the expected dimension of the dual variety of a variety with degenerate Gauss map appeared also in the paper [Pio 2b] by Piontkowski and implicitly in the books Landsberg [L 99] (see 7.2.1.1 and 7.3i) and Fischer and Piontkowski [FP 01] (Section 2.3.4).

During the last 20 years, the smooth dually degenerate varieties (for which $\dim X^* < N - 1$) were considered in many articles (see, for example, the papers [GH 79] by Griffiths and Harris, Zak [Za 87], Ein [E 85, 86] and the books [Ha 92] by Harris, [L 99] by Landsberg, [T 01] by Tevelev, [FP 01] by Fisher and Piontkowski). Note that Harris [Ha 92] (p. 197) uses the term *deficient* for such varieties and the term *deficiency* for their defect.

The classification of dually degenerate smooth varieties of small dimensions n with positive dual defect δ_* was found by Ein [E 85, 86] for $n \leq 6$, by Ein [E 85, 86] and Lanteri and Strupa [LS 87] for $n = 7$, and by Beltrametti, Fania, and Sommese [BFS 92] for $n \leq 10$ (see also Section 9.2.C in the book [T 01] by Tevelev).

For applications of the duality principle see also the book [AG 93] by Akivis and Goldberg. In our presentation of these applications, we follow our recently published papers [AG 01a, 02b] and Section 4.1 of the book [AG 93].

The dual defect of a variety X must be defined as the difference between an expected dimension of the dual variety X^* and its true dimension. Thus, the definition given on p. 71 and used in the literature (see, for example, Fischer and Piontkowski [FP 01] (p. 55); Harris [Ha 92] (p. 199); Landsberg [L 99] (p. 16); and Tevelev [T 01] (p. 3) is correct for smooth varieties because for them an expected dimension of the dual variety X^* equals $N - 1$. In the books mentioned above, the definitions of the dual defect and dually degenerate varieties given on p. 71, which are correct for tangentially nondegenerate varieties, are automatically extended to varieties with degenerate Gauss maps. In our opinion, this is incorrect, because for the latter varieties, an expected dimension of X^* is $N - l - 1 < N - 1$ (see formula (2.65)), and for them the correct definition of the dual defect (and dually degenerate varieties) must be the definitions given on p. 72. Note that the definition on p. 72 includes the definition on p. 71: the latter can be obtained from the former if one takes $l = 0$. Note also that by definition on p. 72, the dual defect δ_* of a dually nondegenerate variety equals 0 (and this is natural), while by the definition on p. 71, $\delta_* = \delta_\gamma = n - r > 0$.

2.6. The constructions we made in Section 2.6 can be generalized for the projective space $\mathbb{KP}$ over the algebras $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, where $\mathbb{C}$ is the algebra of complex numbers, $\mathbb{H}$ is the algebra of quaternions, and $\mathbb{O}$ is the algebra of Cayley's octonions or octaves (see more on octonions and the algebra of Cayley's octonions in Rosenfeld [Ro 97], Section 1.3.1). Then $\dim \mathbb{K} = 2^{i-1}$, $i = 1, 2, 3, 4$. In all these algebras, there is an involutive or antiinvolutive automorphism $z \to \overline{z}$.

This was done by Ishikawa in [I 99a], who constructed examples of real algebraic cubic nonsingular hypersurfaces with degenerate Gauss maps in $\mathbb{RP}^n$ for $n = 4, 7, 13, 25$. These hypersurfaces have the structure of homogeneous spaces of groups $\mathbf{SO}(3), \mathbf{SU}(3), \mathbf{Sp}(3)$, and F_4, respectively, and their projective duals are linear projections of Veronese embeddings of projective planes $\mathbb{KP}^2$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

Chapter 3

Basic Equations of Varieties with Degenerate Gauss Maps

In Section 3.1, we define the Monge–Ampère foliation associated with a variety with a degenerate Gauss map of dimension n, derive the basic equations of varieties with degenerate Gauss maps, and prove a characteristic property of such varieties (the Monge–Ampère foliation has flat leaves) of any of their plane generators. At the end of Section 3.1, for varieties with degenerate Gauss maps we prove the generalized Griffiths–Harris Theorem, which becomes the well-known Griffiths–Harris Theorem for tangentially nondegenerate varieties (see the paper [GH 79] by Griffiths and Harris). In Section 3.2, we consider focal images of such varieties (the focus hypersurfaces and the focus hypercones). In Section 3.3, we study varieties with degenerate Gauss maps without singularities, in Section 3.4, we introduce and investigate an important class of varieties with degenerate Gauss maps without singularities, the so-called Sacksteder–Bourgain hypersurface, in the affine space $\mathbb{A}^4$, and in Section 3.5, we consider complete parabolic varieties in Riemannian spaces of constant curvature.

3.1 The Monge–Ampère Foliation

3.1.1 The Monge–Ampère Foliation Associated with a Variety with a Degenerate Gauss Map. We consider a variety X with a degenerate Gauss map of dimension n and rank r in the space $\mathbb{P}^N$. Let

$$\gamma : X \to \mathbb{G}(n, N)$$

be its Gauss map. Denote by L a leaf of the Gauss map γ. This leaf is the preimage of the tangent subspace $T_x(X)$ on the variety X:

$$L = \gamma^{-1}(T_x) = \gamma^{-1}(\gamma(x)).$$

The foliation on X defined as indicated above is called the *Monge–Ampère foliation* (see, for example, the papers by Delanoë [De 89] and Ishikawa [I 98, 99b]).

A leaf L of this foliation, as well as the tangent subspace $T_x(X)$, depends on r parameters. Denote by M an r-dimensional variety of parameters defining a displacement of the subspace T_x on X, and let $(u^{l+1}, \dots, u^n)$ be coordinates of a point of M.

The Monge–Ampère foliation is defined by a completely integrable system of Pfaffian equations

$$\omega^p = 0, \quad p = l+1, \dots, n,$$

whose first integrals are coordinates $(u^{l+1}, \dots, u^n)$ of a point $u \in M$.

Because the 1-forms ω_i^α occurring in equations (2.40) define a displacement of the subspace T_x on X, on the variety X, these forms must be expressed in terms of precisely r linearly independent forms, i.e., we have

$$\operatorname{rank}\,(\omega_i^\alpha) = r.$$

If $x = A_0$ is a regular point of the variety X, then we can take as these independent forms the forms

$$\omega_0^p = \omega^p, \ \ p = l+1, \dots, n,$$

determining a displacement of the point x transversally to the leaf L_x of the Monge–Ampère foliation. These forms are basis forms on the manifold M and on the Monge–Ampère foliation of the variety X. They are linear combinations of the differentials du^p of coordinates of a point $u \in M$.

3.1.2 Basic Equations of Varieties with Degenerate Gauss Maps. We write the expressions of the forms ω_i^α in terms of the forms ω^q, $q = l+1, \dots, n$, as follows:

$$\omega_i^\alpha = b_{iq}^\alpha \omega^q, \quad q = l+1, \dots, n. \tag{3.1}$$

Because the matrix (b_{ij}^α), $i, j = 1, \dots, n$, is symmetric, this matrix takes the form

$$\begin{pmatrix} O_{l\times l} & O_{l\times r} \\ O_{r\times l} & (b_{pq}^\alpha) \end{pmatrix}, \ \ b_{pq}^\alpha = b_{qp}^\alpha, \tag{3.2}$$

where $O_{p\times q}$ is the zero matrix with p rows and q columns. In what follows, we will assume that there is at least one nondegenerate matrix of rank r in the system $(\xi_\alpha b^\alpha_{pq})$ of second fundamental tensors of X. By the generalized Griffiths–Harris theorem (see p. 97), this means that the variety X in question is dually nondegenerate.

In what follows we will use the following ranges of indices:

$$a, b, c = 1, \dots, l; \; p, q = l+1, \dots, n; \; \alpha, \beta = n+1, \dots, N.$$

We will choose the points of our moving frame as follows: the point $A_0 = x$ is a regular point of X; the points A_a belong to the leaf L of the Monge–Ampère foliation passing through the point A_0; the points A_p together with the points A_0, A_a define the tangent subspace $T_L X$ to X; and the points A_α are located outside the subspace $T_L X$.

It follows from equations (2.5), (3.1) and (3.2) that

$$\omega^\alpha = 0, \;\; \omega^\alpha_a = 0, \tag{3.3}$$

$$\omega^\alpha_p = b^\alpha_{pq} \omega^q, \tag{3.4}$$

where, as earlier, $b^\alpha_{pq} = b^\alpha_{qp}$, and the indices take the values indicated above. The 1-forms ω^q are basis forms of the Gauss image $\gamma(X)$ of the variety X, and the quantities b^α_{pq} form the second fundamental tensor of the variety X at the point x.

By (3.3), the equations of infinitesimal displacement of the moving frame associated with a variety X with a degenerate Gauss map have the form

$$\begin{cases} dA_0 = \omega^0_0 A_0 + \omega^a A_a + \omega^p A_p, \\ dA_a = \omega^0_a A_0 + \omega^b_a A_b + \omega^p_a A_p, \\ dA_p = \omega^0_p A_0 + \omega^a_p A_a + \omega^q_p A_q + \omega^\alpha_p A_\alpha, \\ dA_\alpha = \omega^0_\alpha A_0 + \omega^\alpha_a A_a + \omega^q_\alpha A_q + \omega^\beta_\alpha A_\beta, \end{cases} \tag{3.5}$$

where here and in what follows, unless otherwise stated, the indices take the values indicated above.

Taking exterior derivatives of equations (3.3), we obtain the following exterior quadratic equations:

$$\omega^p_a \wedge \omega^\alpha_p = 0. \tag{3.6}$$

Substituting expressions (3.4) into equations (3.6), we find that

$$b^\alpha_{pq} \omega^p_a \wedge \omega^q = 0. \tag{3.7}$$

Let us prove that, as was the case for the forms ω^α_p, the forms ω^p_a can be expressed in terms of the basis forms ω^q alone. Suppose that decompositions of the forms ω^p_a have the following form:

$$\omega^p_a = c^p_{aq} \omega^q + c^p_{a\xi} \omega^\xi, \tag{3.8}$$

where the forms ω^ξ are 1-forms that are linearly independent themselves, and the set $\{\omega^q, \omega^\xi\}$ is also linearly independent. Substituting these expressions into equations (3.7), we find that

$$b^\alpha_{sq} c^s_{ap} \omega^p \wedge \omega^q + b^\alpha_{pq} c^p_{a\xi} \omega^\xi \wedge \omega^q = 0.$$

Because the exterior products $\omega^p \wedge \omega^q$ and $\omega^\xi \wedge \omega^q$ are independent, it follows from these relations that

$$\boxed{b^\alpha_{sq} c^s_{ap} = b^\alpha_{sp} c^s_{aq}} \tag{3.9}$$

and

$$b^\alpha_{pq} c^p_{a\xi} = 0. \tag{3.10}$$

The last system is a linear homogeneous system with respect to the quantities $c^p_{a\xi}$. Because the forms ω^α_p cannot be expressed in terms of less than r linearly independent forms, the rank of the matrix of coefficients of system (3.10) is equal to r, and the system has only the trivial solution, $c^p_{a\xi} = 0$. Hence equations (3.8) take the form:

$$\omega^p_a = c^p_{aq} \omega^q, \tag{3.11}$$

where the coefficients c^p_{aq} are connected with the coefficients b^α_{pq} by conditions (3.9). Note that conditions (3.9) will play an important role in our future investigations, and we will apply them many times. For this reason, we framed these conditions.

We shall call equations (3.4) and (3.11) the *basic equations* of a variety X with a degenerate Gauss map.

Note that under transformations of the points A_p, the quantities c^p_{aq} are transformed as tensors. As to the index a, the quantities c^p_{aq} do not form a tensor with respect to this index. Nevertheless, under transformations of the points A_0 and A_a, the quantities c^p_{aq} along with the unit tensor δ^p_q are transformed as tensors. For this reason, the system of quantities c^p_{aq} is called a *quasitensor*.

Denote by B^α and C_a the $(r \times r)$-matrices of coefficients occurring in equations (3.4) and (3.11):

$$B^\alpha = (b^\alpha_{pq}), \quad C_a = (c^p_{aq}).$$

Sometimes we will use the identity matrix $C_0 = (\delta^p_q)$ and the index $i = 0, 1, \ldots, l$, i.e., $\{i\} = \{0, a\}$. Then equations (3.2) and (3.9) can be combined and written as follows:

$$(B^\alpha C_i)^T = (B^\alpha C_i), \tag{3.12}$$

i.e., the matrices

$$H^\alpha_i = B^\alpha C_i = (b^\alpha_{qs} c^s_{ip})$$

are symmetric.

3.1.3 The Structure of Leaves of the Monge–Ampère Foliation. In examples considered in Section 2.4 we saw that the leaves of the Monge–Ampère foliation are straight lines or planes. The following theorem proves that this is the general fact.

Theorem 3.1. *The leaves of the Monge–Ampère foliation associated with a variety X with a degenerate Gauss map of dimension n and rank r are l-dimensional planes, where $l = n - r$.*

Proof. On a variety X, consider the system of equations

$$\omega^p = 0, \tag{3.13}$$

defining the Monge–Ampère foliation. By (2.6) and (3.11), we have

$$d\omega^p = \omega^q \wedge (\omega_q^p - \delta_q^p \omega_0^0 - c_{aq}^p \omega^a). \tag{3.14}$$

By the Frobenius theorem, equations (3.14) imply that the system of equations (3.13) is completely integrable and defines a foliation of the variety X into $(n-r)$-dimensional varieties.

Let us prove that all these varieties are planes. In fact, if $\omega^p = 0$, then by (3.11), the first two of equations (3.5) take the form

$$\begin{aligned} dA_0 &= \omega_0^0 A_0 + \omega^a A_a, \\ dA_a &= \omega_a^0 A_0 + \omega_a^b A_b. \end{aligned} \tag{3.15}$$

This means that if $\omega^p = 0$, then the l-plane defined by the points A_0 and A_a remains constant. Thus the varieties defined on X by the system of equations (3.13) are planes of dimension l. Namely, these l-planes are the leaves of the Monge–Ampère foliation associated with X.

The same system of equations (3.13) implies that if $\omega^p = 0$, then in addition to equations (3.15), we have the equations

$$dA_p = \omega_p^0 A_0 + \omega_p^a A_a + \omega_p^q A_q. \tag{3.16}$$

Equations (3.15) and (3.16) mean that the tangent subspace $T_x(X)$ along the fixed l-plane $A_0 \wedge A_1 \wedge \ldots \wedge A_l$ remains constant. □

Thus, in Theorem 3.1, we have proved that the leaves of the Monge–Ampère foliation associated with the variety X are l-dimensional planes

$$L = A_0 \wedge A_1 \wedge \ldots \wedge A_l$$

or their open parts. The tangent subspace $T_x(X)$ remains constant along the leaves L of this foliation. For this reason, it is natural to denote this subspace by T_L, $L \subset T_L$. A pair (L, T_L) on X depends on r parameters $u^{l+1}, \dots, u^n$, the coordinates of the point $u \in M$.

In what follows, *we extend the leaves L of the Monge–Ampère foliation to the entire projective space $\mathbb{P}^l$ in which they are located* ($L \sim \mathbb{P}^l$). The leaves $L \sim \mathbb{P}^l$ of the Monge–Ampère foliation are *plane generators* of the variety X. Note that the Monge–Ampère foliation is not locally trivial because its leaves have singular points (which we will consider in Section 3.2).

We shall call the varieties of this type *projectively complete*. The notion of geodesic completeness, used when one studies varieties with degenerate Gauss maps in the Euclidean geometry, cannot be used here because in projective geometry the geodesics on submanifolds cannot be defined. In a certain sense, the notion of projective completeness replaces the notion of geodesic completeness in the Euclidean (and Riemannian) geometry.

However, unlike a traditional definition of the foliation (see, for example, Dubrovin, Fomenko, and Novikov [DFN 85], §29), as we will see in Section 3.2, the leaves of the Monge–Ampère foliation can have singularities. It is for this reason that in general its leaves are not diffeomorphic to a standard leaf $\mathbb{P}^l$.

3.1.4 The Generalized Griffiths–Harris Theorem. In Section 2.5.1, we defined the dual variety $X^* \subset (\mathbb{P}^N)^*$ for a variety $X \subset \mathbb{P}^N$ with a degenerate Gauss map of dimension n and rank r as the set of tangent hyperplanes $\xi\,(\xi \supset T_L X)$ to X. It follows that the dual variety X^* is a fibration whose fiber is the bundle

$$\Xi = \{\xi | \xi \supset T_L X\}$$

of hyperplanes ξ containing the tangent subspace $T_L X$ and whose base is the manifold

$$B = X^*/\Xi.$$

As we noted in Section 2.5.1, the dimension of a fiber Ξ of this fibration equals $N - n - 1$, and the dimension of the base B equals r, $\dim B = r$, i.e., the dimension of B coincides with the rank of the variety X. This implies that in the general case,

$$\dim X^* = (N - n - 1) + r = N - l - 1$$

(cf. formula (2.65)).

As was noted in Section 2.5.1, for a dually degenerate variety X with a degenerate Gauss map, we have

$$\dim X^* < N - l - 1.$$

The following theorem expresses this condition in terms of the second fundamental forms of the variety X.

Theorem 3.2 (Generalized Griffiths–Harris Theorem). *The dual variety $X^* \subset (\mathbb{P}^N)^*$ of a variety X with a degenerate Gauss map is dually degenerate if and only if at all smooth points $x \in X$ every second fundamental form of the system of second fundamental forms $II = \xi_\alpha b^\alpha_{pq} \omega^p \omega^q$ of X is degenerate.*

Proof. Consider the bundle $\mathcal{R}(X)$ of frames associated with a variety X with a degenerate Gauss map, which we constructed earlier in this section. The basis forms of the bundle $\mathcal{R}(X)$, as well as the basis forms of the tangent bundle $T(X)$ and the Monge–Ampère foliation of the variety X, are also called the *horizontal forms*, and the secondary forms of all these bundles are called the *fiber* or *vertical forms* (see, for example, Section 20.2 of the book [Di 71] by Dieudonné). The forms ω^p, $p = l+1, \ldots, n$, are linearly independent, and their number equals r. Thus, these forms are basis forms in the bundle $\mathcal{R}(X)$. On the bundle $\mathcal{R}(X)$ the equations of infinitesimal displacement of a frame have the form (3.5).

In this proof we will use the following ranges of indices:

$$\begin{array}{ll} 0 \leq u, v \leq N, & 1 \leq i, j \leq n, \\ 1 \leq a, b \leq l, & l+1 \leq p, q \leq n, \\ n+1 \leq \alpha, \beta \leq N, & n+1 \leq \rho, \sigma \leq N-1. \end{array}$$

Consider now the dual coframe (or tangential frame) $\{\alpha^u\}$ in the space $(\mathbb{P}^N)^*$ to the frame $\{A_u\}$ constructed in Section 1.3. The hyperplanes α^u of the frame $\{\alpha^u\}$ are connected with the points of the frame $\{A_u\}$ by the conditions (see equations (1.77)):

$$(\alpha^u, A_v) = \delta^u_v. \tag{3.17}$$

Conditions (3.17) mean that the hyperplane α^u contains all points A_v, $v \neq u$, and that the condition of normalization $(\alpha^u, A_u) = 1$ (cf. formula (1.70)) holds.

We proved in Section 1.3 that the equations of infinitesimal displacement of the tangential frame $\{\alpha^u\}$ are

$$d\alpha^u = \widetilde{\omega}^u_v \alpha_v, \quad u, v = 0, 1, \ldots, N, \tag{3.18}$$

where the forms $\widetilde{\omega}^v_u$ are related to the forms ω^v_u by the following formulas:

$$\widetilde{\omega}^u_v = -\omega^u_v$$

(see equations (1.78)). Hence equations (3.18) can be written as

$$d\alpha^u = -\omega_v^u \alpha^v \tag{3.19}$$

(see equations (1.79)).

Recalling that

$$\begin{cases} dA_0 \equiv \omega^a A_a + \omega^p A_p \pmod{A_0}, \\ dA_a \equiv \omega_a^p A_p \pmod{A_0, A_1, \ldots, A_l}, \\ dA_p \equiv \omega_p^\sigma A_\sigma + \omega_p^N A_N \pmod{A_0, A_1, \ldots, A_n} \end{cases}$$

(cf. equations (3.5)) and

$$\omega^\alpha = 0, \;\; \omega_a^\alpha = 0, \;\; \omega_p^\alpha = b_{pq}^\alpha \omega^p$$

(cf. equations (2.5), (3.3), and (3.4)), it follows from (3.19) that

$$d\alpha^N \equiv -\omega_a^N \alpha^a - \omega_p^N \alpha^p - \omega_\sigma^N \alpha^\sigma \pmod{\alpha^N}.$$

The $N-n-1$ forms ω_σ^N determine the infinitesimal displacement of the hyperplane $\xi = \alpha^N$ in the bundle Ξ of tangent hyperplanes ξ containing the tangent subspace $T_L X$, i.e., these forms are the fiber forms on the dual variety X^*. The number $N - n - 1$ coincides with the dimension of a fiber of this bundle. Hence, forms ω_σ^N are linearly independent.

A basis of the fibration X^* is the span S^N of the forms ω_p^N, i.e., the forms ω_p^N are horizontal on X^*. Because

$$\omega_p^N = b_{pq}^N \omega^q, \;\; b_{pq}^N = b_{qp}^N, \;\; p, q = l+1, \ldots, n,$$

the dimension of S^N does not exceed the rank $r = n - l$ of the variety X.

Consider the exterior product

$$\omega_{l+1}^N \wedge \ldots \wedge \omega_n^N = \det (b_{pq}^N)\, \omega^{l+1} \wedge \ldots \wedge \omega^n.$$

It is easy to see that $\dim S^N = \operatorname{rank} (b_{pq}^N)$, and $\dim S^N < r$ if and only if $\det (b_{pq}^N) = 0$.

Because α^N was any of the hyperplanes α^β, we have

$$\det (b_{pq}^\beta) = 0.$$

Moreover, the tangent hyperplane ξ can be chosen arbitrarily from the system $\xi = \xi_\beta \alpha^\beta$. This system of tangent hyperplanes passing through the tangent subspace $T_L X$ determines the system of second fundamental forms

$$II = \xi_\beta b_{pq}^\beta \omega^p \omega^q$$

of the variety X and the system of second fundamental tensors

$$\xi_\beta b^\beta_{pq}$$

of this variety X. This proves the theorem statement: the variety X is dually degenerate if and only if the system of its second fundamental forms II does not contain any nondegenerate form. □

Note that *if* $r = n$ (*i.e., if a variety* X *is tangentially nondegenerate*), *then its dual variety* X^* *is dually degenerate if and only if at all smooth points* $x \in X$ *every second fundamental form of the system of second fundamental forms* $II = \xi_\alpha b^\alpha_{ij}\omega^i\omega^j$ *of* X *is degenerate.* We emphasize that unlike in Theorem 3.2, the basis forms here are the forms ω^i, $i = 1, \dots, n$.

This is exactly Theorem 3.5 proved by Griffiths and Harris in [GH 79]. This is why we called Theorem 3.2 the *generalized Griffiths–Harris theorem.*

Corollary 3.3. *A variety* X *with a degenerate Gauss map is dually nondegenerate* (*i.e., the dimension of its dual variety* $X^* \subset (\mathbb{P}^N)^*$ *equals* $N - l - 1$) *if and only if at any smooth point* $x \in X$ *there is at least one nondegenerate second fundamental form in the system of second fundamental forms* $\xi_\alpha b^\alpha_{pq}\omega^p\omega^q$ *of* X*.*

3.2 Focal Images

3.2.1 The Focus Hypersurfaces. Let $X = V^n_r$ be a variety with a degenerate Gauss map of rank r in the space $\mathbb{CP}^N$. By Theorem 3.1, such a variety carries an r-parameter family of l-dimensional plane generators L of dimension $l = n - r$. Let $x = x^0 A_0 + x^a A_a$ be an arbitrary point of a generator L. For such a point we have

$$dx = (dx^0 + x^0\omega^0_0 + x^a\omega^0_a)A_0 + (dx^a + x^0\omega^a + x^b\omega^a_b)A_a + (x^0\omega^p + x^a\omega^p_a)A_p.$$

By (3.11), it follows that

$$dx \equiv (x^0\delta^p_q + x^a c^p_{aq})A_p\omega^q \pmod{L}. \tag{3.20}$$

The matrix $(J^p_q) = (x^0\delta^p_q + x^a c^p_{aq})$ is the *Jacobi matrix* of the map $\gamma : X \to \mathbb{G}(n, N)$, and the determinant

$$J(x) = \det(J^p_q) = \det(x^0\delta^p_q + x^a c^p_{aq})$$

of this matrix is the *Jacobian* of the map γ.

We recall that in Section 2.1.1 we call a point $x \in X$ a regular point of the map f and of the variety X if $\dim T_x X = \dim X = n$, and we call $x \in X$

singular if $\dim T_x X > \dim X = n$. It is easy to see that at regular points $J(x) \neq 0$ and at singular points $J(x) = 0$. The set of all singular points of the variety X was denoted by $\operatorname{Sing} X$.

By (3.20), the set of singular points of a generator L of the variety X is determined by the equation

$$\det(\delta^p_q x^0 + c^p_{aq} x^a) = 0. \tag{3.21}$$

Hence this set is an *algebraic hypersurface in the generator L of dimension $l-1$ and degree r.* This hypersurface (in L) is called the *focus hypersurface*[1] and is denoted F_L. Obviously, we have $\operatorname{Sing} X = \cup F_L$.

Note that extending the leaves L of the Monge–Ampère foliation to the space $\mathbb{P}^l$ (see Section 3.1) means that we include singular points F_L of L into L. Essentially, by adding F_L to L, we consider the closure of the leaf L, and this closed leaf carries the structure of the projective space $\mathbb{P}^l$.

Because for $x^a = 0$ the left-hand side of equation (3.21) takes the form

$$\det(x^0 \delta^p_q) = (x^0)^r,$$

it follows that the point A_0 is a regular point of the generator L.

We now calculate the second differential of a point $x \in L$:

$$d^2 x \equiv \omega^\alpha_s (\omega^s x^0 + \omega^s_a x^a) A_\alpha \pmod{T_x}.$$

This expression is the *second fundamental form* of the variety X:

$$II_x = \omega^\alpha_s (\omega^s x^0 + \omega^s_a x^a) A_\alpha = b^\alpha_{ps} (\delta^s_q x^0 + c^s_{aq} x^a) A_\alpha \omega^p \omega^q. \tag{3.22}$$

Theorem 3.4. *The number of linearly independent forms in the system of second fundamental forms of a variety X with a degenerate Gauss map of rank r is constant at all regular points of its plane generator L.*

Proof. Suppose that $\xi = \xi_\alpha x^\alpha = 0$ is the tangent hyperplane to X at $x \in L$, $\xi \supset T_L$. Then

$$(\xi, II_x) = h_{pq}(\xi, x) \omega^p \omega^q,$$

where

$$h_{pq}(\xi, x) = \xi_\alpha b^\alpha_{ps} (\delta^s_q x^0 + c^s_{aq} x^a), \quad h_{pq} = h_{qp},$$

is the *second fundamental form* of the variety X at x with respect to the hyperplane ξ. Because at regular points $x \in L$ the inequality $J(x) \neq 0$ holds, the rank of the matrices $(h_{pq}(\xi, x))$ is the same as the rank of the matrix

$$B(\xi) = (\xi_\alpha b^\alpha_{pq}) = \xi_\alpha B^\alpha, \tag{3.23}$$

[1] We use the term "focus hypersurface" for the locus of foci in a plane generator L of a variety X. We will use the term "focal variety" for the locus of foci in the entire variety X.

and this rank is the same at all regular points $x \in L$. Denote this rank by m. □

3.2.2 The Focus Hypercones. We call a tangent hyperplane $\xi = (\xi_\alpha)$ *singular* (or a *focus hyperplane*) if

$$\det(\xi_\alpha b^\alpha_{pq}) = 0, \tag{3.24}$$

i.e., if the rank of matrix (3.23) is reduced. Condition (3.24) is an equation of degree r with respect to the tangential coordinates ξ_α of the hyperplane ξ containing $T_L(X)$. Because we assume that the variety X is dually nondegenerate, by Corollary 3.3, *there exists at least one nondegenerate form in the system of second fundamental forms of* X. Hence in the space $\mathbb{P}^N$, equation (3.24) defines an algebraic hypercone of degree r, whose vertex is the tangent subspace $T_L(X)$. This hypercone is called the *focus hypercone* and is denoted Φ_L (see Akivis and Goldberg [AG 93], p. 119). In the dual space $(\mathbb{P}^N)^*$, equations (3.24) define an algebraic hypersurface Φ^* of degree r belonging to the leaf L^* of the Monge–Ampère foliation on the variety X^* dual to X.

Note that if a variety X is dually degenerate, then on such a variety, equations (3.24) are satisfied identically, and X does not have focus hypercones.

The determinant $\det(\xi_\alpha b^\alpha_{pq})$ on the left-hand side of equation (3.24) is the Jacobian of the dual map $\gamma^* : X^* \to \mathbb{G}(r, N)$. The map γ^* sends a hyperplane ξ tangent to the variety X (i.e., an element of the variety X^*) to a leaf L of the Monge–Ampère foliation that belongs to the hyperplane ξ.

The focus hypersurface $F_L \subset L$ (defined in Section 3.2.1) and the focus hypercone Φ_L with vertex T_L are called the *focal images* of the variety X with a degenerate Gauss map.

Note that under the passage from the variety $X \subset \mathbb{P}^N$ to its dual variety $X^* \subset (\mathbb{P}^N)^*$, the systems of square matrices C_a and B^α as well as the focus hypersurfaces F_L and the focus cones Φ_L exchange their roles.

Because

$$d^2 x \equiv b^\alpha_{qs}(\delta^s_p x^0 + c^s_{ap} x^a)\omega^p \omega^q A_\alpha \pmod{T_L, x \in L},$$

the points

$$A_{pq} = b^\alpha_{qs}(\delta^s_p x^0 + c^s_{ap} x^a) A_\alpha, \;\; A_{pq} = A_{qp}, \tag{3.25}$$

together with the points A_0, A_a, and A_p, define the osculating subspace $T^2_L(X)$. Its dimension is

$$\dim T^2_L(X) = n + m,$$

where m is the number of linearly independent points among the points A_{pq}, $m \leq \min\{\frac{r(r+1)}{2}, N - n\}$. The number m is the number of linearly independent scalar second fundamental forms of the variety X at its regular points. However, because at a regular point $x \in X_{sm}$ the condition $J(x) \neq 0$ holds (see p. 97), m is the number of linearly independent points among the points

$$\widetilde{A}_{pq} = A_\alpha b^\alpha_{pq}.$$

The number m is constant for all regular points of a generator L of the variety X. We also use the notation S_L for the osculating space $T^2_L(X)$.

On a generator L of the variety X, consider the system of equations

$$\delta^q_p x^0 + c^q_{ap} x^a = 0. \tag{3.26}$$

The matrix of system (3.26) has r^2 rows and $l+1$ columns. Denote the rank of this matrix by m^*. If $m^* < l+1$, then system (3.26) defines a subspace K_L of dimension $k = l - m^*$ in L. This subspace belongs to the focus hypersurface F_L defined by equation (3.21). If $l > m^*$, then the hypersurface F_L becomes a cone with vertex K_L. We call the subspace K_L the *characteristic subspace* of the generator L.

Note also that by the duality principle in $\mathbb{P}^N$, the osculating subspace S_L and the characteristic subspace K_L constructed for a pair (L, T_L) correspond to one another.

In what follows, we assume that a variety X in question does not have singular points except the foci determined by equation (3.21), and its dual X^* does not have singular hyperplanes except the focus hyperplanes determined by equation (3.24).

3.2.3 Examples. First we will find the foci and the focus hypersurfaces F_L for some of examples considered in Section 2.3.

Example 2.4. For a cone X, the focus hypersurface F_L in each its plane generators L is the cone vertex S.

Example 2.5. For a torse, in the case $n = 2, r = 1$ (i.e., for $X = V^2_1 \subset \mathbb{P}^N$) formed by the tangents to a nonplanar curve in $\mathbb{P}^N$ (see Sections 2.2.4 and Figure 2.3 for $N = 3$), each of its rectilinear generators L has one singular point (a focus), and the tangent subspace of X at this point degenerates into a straight line. The set of all singular points of such X forms the *edge of regression* of this variety with a degenerate Gauss map.

Note that in Examples 2.4 and 2.5 the focus variety F_L of the generator L is a subspace of dimension $l-1$. However, for a cone in Example 2.4, F_L is fixed, while for a torse in Example 2.5, F_L depends on one parameter and is itself a torse of dimension one.

Example 2.6. For a join, the points y_1 and y_2 are foci of the generator L, and the curves Y_1 and Y_2 are degenerate focus varieties (see Figure 2.4 on p. 69).

There are two cones through every generator L. These cones are described by generators passing through the focus y_1 or the focus y_2. On the variety X, these cones form two one-parameter families comprising a focal net of the variety X.

Example 2.7. For a hypersurface $X = V_2^3 \subset \mathbb{P}^4$ on Figure 2.5 (the case $n = 3$, $r = 2$), the focus hypersurface F_L in each of its one-dimensional plane generators L is decomposed into a pair of points y_1 and y_2 called the foci. These foci y_1 and y_2 describe two two-dimensional focal surfaces Y_1 and Y_2. Note that Example 2.6 differs from Example 2.7 because in Example 2.6 the foci describe the curves while in Example 2.7 they describe two-dimensional surfaces.

Example 2.9. For the cubic symmetroid defined by equation (2.71), the focus curve F_L in a two-dimensional generator L is a conic (see formula (2.79) on p. 79). The manifold of these curves on the symmetroid (2.71) is a Veronese variety defined in the space $(\mathbb{P}^5)^*$ by the equation

$$\operatorname{rank}\ (a_{ij}) = 1.$$

Example 3.5. (See also Section 5.3, p. 184.) Let X be a variety with a degenerate Gauss map of dimension n and rank $r = n - 1$. The leaves of the Monge–Ampère foliation on X can be considered as light rays. The focus hypersurfaces $F_L \subset L$ decompose into r points on L. Each of these points describe an n-dimensional variety $F_{(p)}$, $p = 1, \ldots, r$. From the point of view of geometric optics, the varieties $F_{(p)}$ are the loci of condensation of light rays. They are called the *caustics* (see, for example, §2 of Chapter 2 in the book [AVGL 89] by Arnol'd, Vasil'ev, Goryunov, and Lyashko). Of course, only the cases $N = 2$ and $N = 3$ are discussed in optics.

Although in this book we assume that $n < N$, the case $n = N$ is also interesting. Then a variety X is the *congruence* (i.e., an $(n-1)$-parameter family) of straight lines. The congruence of straight lines in a three-dimensional projective, Euclidean, and non-Euclidean spaces were studied in detail by many geometers starting from Monge [Mon 50] (for a detailed theory of congruences see the book [Fi 50] by Finikov).

3.2.4 The Case $n = 2$. We now consider the case $n = 2$, i.e., $X^2 \subset \mathbb{P}^N$. In this case, we have $0 \leq r \leq 2$. As we already know, if $r = 2$, then X is a tangentially nondegenerate smooth surface. If $r = 0$, then X is $\mathbb{P}^2$. In this case, $B^\alpha = 0$ and $C_i = 0$, and there are no singularities (the focus hypersurfaces F_L and the focus hypercones Φ_L are indeterminate).

Suppose that $r = 1$. Because in each of the pencils $\xi_\alpha B^\alpha$ and $\xi^i C_i$, there is at least one nondegenerate matrix, and $C_0 = (\delta_q^p)$, we have rank $B^\alpha = 1$, and rank $C_a \leq 1$.

In this case, the generator L depends on one parameter. The indices p, q take only one value, 2, and equation (3.21) of the focus hypersurface F_L takes the form

$$x^0 + c^2_{12} x^1 = 0.$$

Thus, in this case the focus hypersurface F_L is a point. Equation (3.24) of the focus hypercone Φ_L takes the form

$$\xi_\alpha b^\alpha_{22} = 0.$$

Thus, in this case the focus hypercone Φ_L is a hyperplane.

In general, the surface X^2 is a torse. If the above mentioned point is held fixed, i.e., it is the same for all rectilinear generators L, then the surface X^2 of rank one is a cone.

3.2.5 The Case $n = 3$. In this case, we have $0 \leq r \leq 3$. As was in the previous case, if $r = 3$, then X^3 is a tangentially nondegenerate smooth hypersurface, and if $r = 0$, then X is $\mathbb{P}^3$.

Suppose first that $r = 2$, and as a result, $l = 1$. Because the variety X is dually nondegenerate, in each of the pencils $\xi_\alpha B^\alpha$ and $\xi^i C_i$, there is at least one nondegenerate matrix, and $C_0 = (\delta^p_q)$, we have rank $B^\alpha = 2$, and rank $C_a \leq 2$. In this case, the rectilinear generator L depends on two parameters. The indices p and q take only two values, 2 and 3, and equation (3.21) of the focus hypersurface F_L takes the form

$$\det (\delta^q_p x^0 + c^q_{1p} x^1) = 0, \quad p, q = 2, 3,$$

i.e., it has the form

$$\begin{vmatrix} x^0 + c^2_{12} x^1 & c^3_{12} x^1 \\ c^2_{13} x^1 & x^0 + c^3_{13} x^1 \end{vmatrix} = 0.$$

This equation defines two foci. Denote them by F_1 and F_2. We present here a complete classification of three-dimensional varieties X for which $F_1 \neq F_2$. A classification of varieties X with a double focus, i.e., when $F_1 = F_2$, will be given in Section 4.5.2.

If $F_1 \neq F_2$, then the following cases are possible:

a) The points F_1 and F_2 describe two-dimensional surfaces (F_1) and (F_2), and the rectilinear generators L are tangent to (F_1) and (F_2) along the lines composing conjugate nets on (F_1) and (F_2) (see Figure 2.5 on p. 70).

b) The points F_1 and F_2 describe the same irreducible two-dimensional surface (F) having two components (F_1) and (F_2). The rectilinear generators L are tangent to the components (F_1) and (F_2).

c) The point F_1 describes a two-dimensional surface (F_1), the point F_2 describes a curve (F_2), and the rectilinear generators $L = F_1F_2$ are tangent to the surface (F_1) and intersect the curve (F_2). Then the variety X foliates into ∞^1 cones with vertices on the curve (F_2), and the surface (F_1) has a conic conjugate net.

d) Both points F_1 and F_2 describe curves (F_1) and (F_2), and the rectilinear generators L intersect both of these curves (see Figure 2.4 on p. 69). Then the variety X foliates into two families of cones whose vertices describe the curves (F_1) and (F_2). So, X is a join (see Example 2.6 in Section 2.4).

e) Both points F_1 and F_2 describe curves (F_1) and (F_2) that are parts of a curve γ not belonging to a three-dimensional space. Then the variety X is described by bisecants to the curve γ.

Note that the case when the point F_1 describes a two-dimensional surface (F_1) in $\mathbb{P}^N$, $N \geq 4$, and the point F_2 is fixed is impossible. If it were possible, then the rectilinear generators $L = F_1F_2$ would be tangent to the surface (F_1) and would pass through the point F_2. But for such a configuration, the point F_1 is not a focus of L.

Suppose next that $r = 1$, and as a result, $l = 2$. In this case, $p, q = 3$, and equation (3.21) of the focus hypersurface F_L takes the form

$$x^0 + c^3_{13}x^1 + c^3_{23}x^2 = 0.$$

Thus, in this case the focus hypersurface F_L is a straight line. Therefore, the variety X foliates into ∞^1 of 2-planes that are osculating planes of a generic curve γ. The focus straight lines F_L are tangent to the curve γ.

In particular, if all F_L have a common point, then the variety X is a cone. This cone is a cone over a developable surface formed by tangents to a curve γ belonging to a three-dimensional subspace.

3.3 Some Algebraic Hypersurfaces with Degenerate Gauss Maps in $\mathbb{P}^4$

The question arises: Do there exist in the space $\mathbb{P}^N$ varieties with degenerate Gauss maps of rank r without singularities? The preceding considerations imply that from the complex point of view, *a variety* $X = V^n_r$ *with a degenerate*

Gauss map of rank r does not have singularities if and only if it is an n-plane $\mathbb{P}^n$, i.e., if $r = 0$. From the real point of view, *a variety $X = V_r^n$ with a degenerate Gauss map of rank r does not have real singularities if and only if its focal images in the plane generators are pure imaginary, and this situation can occur only if the rank r is even.*

Note that in the theory of varieties with degenerate Gauss maps, *the complex point of view is necessary only for studying focal images defined by algebraic equations.*

We consider now two examples: one in which a variety X has real singularities and one in which X does not have real singularities. Other examples will be considered in Chapter 4.

Example 3.6. We consider the hypercubic $X^3 \subset \mathbb{P}^4$ defined by

$$X^3 = \{(x, y, t, w) \in \mathbb{P}^4 \mid w = (x^2 + y^2 + 2txy)/(1 - t^2)\},$$

where (x, y, t, w) are nonhomogeneous coordinates in $\mathbb{P}^4$. Introduce homogeneous coordinates $(x_0, x_1, x_2, x_3, x_4)$ by setting

$$x_1 = xx_0, \quad x_2 = yx_0, \quad x_3 = tx_0, \quad x_4 = wx_0.$$

Then the equation of X^3 becomes

$$F(x_0, x_1, x_2, x_3, x_4) = x_0(x_1^2 + x_2^2) + 2x_1x_2x_3 - x_4(x_0^2 - x_3^2) = 0, \tag{3.27}$$

where x^α, $\alpha = 0, 1, 2, 3, 4$, are the coordinates of a point $x \in \mathbb{P}^4$ with respect to the fixed frame formed by the points

$$E_0(1,0,0,0,0), E_1(0,1,0,0,0), E_2(0,0,1,0,0), E_3(0,0,0,1,0), E_4(0,0,0,0,1).$$

Let us find singular points of the hypercubic X^3. Such points are defined by the equations $\frac{\partial F}{\partial x_\alpha} = 0$, $\alpha = 0, 1, 2, 3, 4$. It follows from (3.27) that

$$\begin{cases} \dfrac{\partial F}{\partial x_0} = x_1^2 + x_2^2 - 2x_0x_4, \\ \dfrac{\partial F}{\partial x_1} = 2x_0x_1 + 2x_2x_3, \\ \dfrac{\partial F}{\partial x_2} = 2x_0x_2 + 2x_1x_3, \\ \dfrac{\partial F}{\partial x_3} = 2x_1x_2 + 2x_3x_4, \\ \dfrac{\partial F}{\partial x_4} = -x_0^2 + x_3^2. \end{cases} \tag{3.28}$$

Equations (3.28) imply that singular points of X^3 are defined by the following system of equations:

$$\begin{cases} x_1^2 + x_2^2 - 2x_0x_4 = 0, \\ x_0x_1 + x_2x_3 = 0, \\ x_0x_2 + x_1x_3 = 0, \\ x_1x_2 + x_3x_4 = 0, \\ -x_0^2 + x_3^2 = 0. \end{cases} \tag{3.29}$$

It follows from the last equation of system (3.29) that

$$x_3 = \pm x_0.$$

If $x_3 = x_0$, then the solution of system (3.29) is

$$\begin{cases} x_3 = x_0, \\ x_2 = -x_1, \\ x_1^2 = x_0x_4, \end{cases} \tag{3.30}$$

and if $x_3 = -x_0$, then the solution of system (3.29) is

$$\begin{cases} x_3 = -x_0, \\ x_2 = x_1, \\ x_1^2 = x_0x_4. \end{cases} \tag{3.31}$$

Systems (3.30) and (3.31) determine two conics C_1 and C_2 belonging to two real 2-planes, π_1 and π_2, defined by the first two equations of (3.30) and (3.31), respectively. A rectilinear generator of X^3 joins two arbitrary points of the conics C_1 and C_2. In fact, take two arbitrary points of C_1 and C_2: $x = (x_0, x_1, -x_1, x_0, x_1^2/x_0) \in C_1$ and $y = (y_0, y_1, y_1, -y_0, y_1^2/y_0) \in C_2$. Consider an arbitrary point $z = x + \lambda y$ on the straight line $x \wedge y$. Its coordinates are

$$z = x + \lambda y = (x_0 + \lambda y_0, x_1 + \lambda y_1, -x_1 + \lambda y_1, x_0 - \lambda y_0, x_1^2/x_0 + \lambda y_1^2/y_0).$$

A straightforward calculation shows that these coordinates satisfy equation (3.27) for any λ. So, the straight line $L = x \wedge y \in X^3$ is a rectilinear generator of X^3. The hypercubic X^3 defined by equation (3.27) is a join with directrices C_1 and C_2 (see Example 2.6), and as a result, X^3 is hypersurface with a degenerate Gauss map of rank two. A tangent hyperplane to X^3 is determined by the points $x \in C_1$ and $y \in C_2$ and the straight lines tangent to C_1 and C_2 at these points (see Figure 3.1).

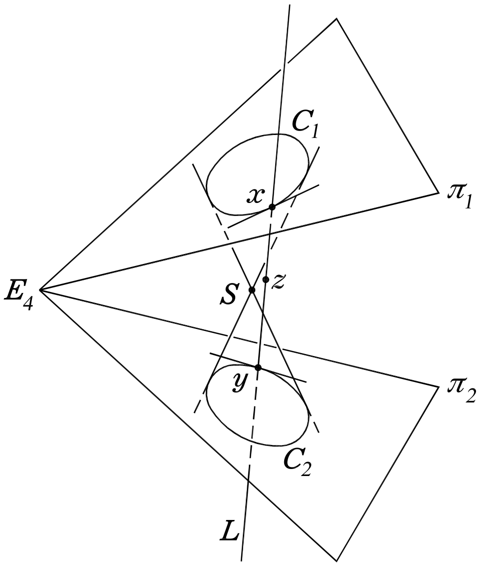


Figure 3.1

Systems (3.30) and (3.31) have the same third equation. This equation determines a second-order hypercone whose vertex is the straight line $S = E_2 \wedge E_3$ defined by the equations $x_0 = x_1 = x_4 = 0$, and the 2-planes π_1 and π_2 do not have common points with the vertex $E_2 \wedge E_3$ of this hypercone. Hence, they intersect this hypercone along two conics C_1 and C_2. Thus, we have a complete description of the hypercubic X defined by equation (3.27).

In this example, we also have $l = 1$ and $m = 1$. The hypercubic X^3 foliates into two family of real cones, whose vertices belong to one of the curves C_1 or C_2 and whose director manifold is the second of these curves.

Let us find the form of the matrices B^α and C_i for this example. We consider the conic C_1 defined by (3.30). It follows from the last equation of (3.30) that

$$\frac{x_1}{x_0} = \frac{x_4}{x_1} = t,$$

and as a result,

$$x_1 = tx_0, \;\; x_4 = t^2 x_0.$$

In addition, equations (3.30) imply that

$$x_3 = x_0, \;\; x_2 = -tx_0.$$

Thus, taking $x_0 = 1$, we obtain a point $x = (1, t, -t, 1, t^2) \in C_1$.

In a similar manner, we find from equations (3.31) of the second conic C_2 that

$$y_1 = sy_0, \;\; y_4 = s^2 y_0, \;\; y_3 = -y_0, \;\; y_2 = sy_0,$$

and taking $y_0 = 1$, we get a point $y = (1, s, s, -1, s^2) \in C_2$.

We choose now as the vertices A_0 and A_1 of our moving frame the following points:

$$\left\{ \begin{array}{l} A_0 = \frac{1}{2}(x + y) = (1, \frac{1}{2}(t + s), \frac{1}{2}(s - t), 0, \frac{1}{2}(t^2 + s^2)), \\ A_1 = \frac{1}{2}(x - y) = (0, \frac{1}{2}(t - s), -\frac{1}{2}(t + s), 1, \frac{1}{2}(t^2 - s^2)). \end{array} \right.$$

Differentiating the points A_0 and A_1, we find that

$$\left\{ \begin{array}{rl} dA_0 &= \frac{1}{2}(0, dt + ds, ds - dt, 0, 2(tdt + sds)) \\ &= \frac{1}{2}(0, 1, -1, 0, 2t)dt + \frac{1}{2}(0, 1, 1, 0, 2s)ds, \\ dA_1 &= \frac{1}{2}(0, dt - ds, -(dt + ds), 0, 2(tdt - sds)) \\ &= \frac{1}{2}(0, 1, -1, 0, 2t)dt + \frac{1}{2}(0, -1, -1, 0, -2s)ds. \end{array} \right.$$

We take the points $\frac{1}{2}(0, 1, -1, 0, 2t)$ and $\frac{1}{2}(0, 1, 1, 0, 2s)$ as the vertices A_2 and A_3 of our moving frame:

$$A_2 = \frac{1}{2}(0, 1, -1, 0, 2t), \;\; A_3 = \frac{1}{2}(0, 1, 1, 0, 2s).$$

Differentiating A_2 and A_3, we obtain

$$dA_2 = (0, 0, 0, 0, 1)dt, \;\; dA_3 = (0, 0, 0, 0, 1)ds.$$

We take the point $(0, 0, 0, 0, 1)$ as the vertex A_4 of our moving frame:

$$A_4 = E_4 = (0, 0, 0, 0, 1).$$

The points A_0, A_1, A_2, A_3, and A_4 are linearly independent. We take them as the vertices of our moving frame.

Thus, for the frame $\{A_0, A_1, A_2, A_3, A_4\}$, we have the following equations of infinitesimal displacement:

$$\left\{ \begin{array}{l} dA_0 = A_2 dt + A_3 ds, \\ dA_1 = A_2 dt - A_3 ds, \\ dA_2 = A_4 dt, \\ dA_3 = A_4 ds, \\ dA_4 = 0. \end{array} \right.$$

Comparing these equations with (3.5) we see that

$$\begin{cases} \omega_0^2 = dt, & \omega_0^3 = ds, \\ \omega_1^2 = dt & \omega_1^3 = -ds, \\ \omega_2^4 = dt, & \omega_3^4 = ds, \\ \omega_4^0 = \omega_4^1 = \omega_4^2 = \omega_4^3 = \omega_4^4 = 0. \end{cases}$$

Comparing equations of the first two rows with (3.11) and the equations of the third row with (3.4), we find that

$$\begin{cases} c_{02}^2 = 1, \quad c_{03}^2 = 0, \quad c_{02}^3 = 0, \quad c_{03}^3 = 1, \\ c_{12}^2 = 1, \quad c_{13}^2 = 0, \quad c_{12}^3 = 0, \quad c_{13}^3 = -1, \\ b_{22}^4 = 1, \quad b_{23}^4 = 0, \quad b_{32}^4 = 0, \quad b_{33}^4 = 1. \end{cases}$$

From the definition of the matrices C_i and B^α, it follows that

$$C_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Equation (3.21) of the focus hypersurface F_L has the form

$$\begin{vmatrix} x_0 + x_1 & 0 \\ 0 & x_0 - x_1 \end{vmatrix} = 0.$$

It follows that $x_1^2 = x_0^2$, i.e., $x_1 = \pm x_0$. Taking $x_0 = 1$, we get $x_1 = \pm 1$. So, the focus hypersurface F_L in the generator $L = A_0 \wedge A_1$ consists of two real points:

$$x = A_0 + A_1 \text{ and } y = A_0 - A_1.$$

Example 3.7. (See Wu and F. Zheng [WZ 02].) We consider the hypercubic $X^3 \subset \mathbb{RP}^4$ defined by

$$X^3 = \{(x, y, t, w) \in \mathbb{P}^4 | w = (x^2 - y^2 + 2txy)/(1 + t^2)\},$$

where (x, y, t, w) are nonhomogeneous coordinates in $\mathbb{RP}^4$. Introduce homogeneous coordinates $(x_0, x_1, x_2, x_3, x_4)$ by setting

$$x_1 = xx_0, \quad x_2 = yx_0, \quad x_3 = tx_0, \quad x_4 = wx_0.$$

Then the equation of X^3 becomes

$$F(x_0, x_1, x_2, x_3, x_4) = x_0(x_1^2 - x_2^2) + 2x_1x_2x_3 - x_4(x_0^2 + x_3^2) = 0. \qquad (3.32)$$

where $x^\alpha,\ \alpha = 0, 1, 2, 3, 4$, are the coordinates of a point $x \in \mathbb{RP}^4$ with respect to the fixed frame formed by the points

$$E_0(1,0,0,0,0), E_1(0,1,0,0,0), E_2(0,0,1,0,0), E_3(0,0,0,1,0), E_4(0,0,0,0,1).$$

Let us find singular points of the hypercubic X^3. Such points are defined by the equations $\frac{\partial F}{\partial x_\alpha} = 0$, $\alpha = 0, 1, 2, 3, 4$. It follows from (3.32) that

$$\begin{cases} \dfrac{\partial F}{\partial x_0} = x_1^2 - x_2^2 - 2x_0x_4, \\ \dfrac{\partial F}{\partial x_1} = 2x_0x_1 + 2x_2x_3, \\ \dfrac{\partial F}{\partial x_2} = -2x_0x_2 + 2x_1x_3, \\ \dfrac{\partial F}{\partial x_3} = 2x_1x_2 - 2x_3x_4, \\ \dfrac{\partial F}{\partial x_4} = -x_0^2 - x_3^2. \end{cases} \tag{3.33}$$

Equations (3.33) imply that singular points of X^3 are defined by the following system of equations:

$$\begin{cases} x_1^2 - x_2^2 - 2x_0x_4 = 0, \\ x_0x_1 + x_2x_3 = 0, \\ -x_0x_2 + x_1x_3 = 0, \\ x_1x_2 - x_3x_4 = 0, \\ -x_0^2 - x_3^2 = 0. \end{cases} \tag{3.34}$$

It follows from the last equation of system (3.34) that

$$x_3 = \pm i x_0.$$

If $x_3 = ix_0$, then the solution of system (3.34) is

$$\begin{cases} x_3 = ix_0, \\ x_2 = ix_1, \\ x_1^2 = x_0x_4, \end{cases} \tag{3.35}$$

and if $x_3 = -ix_0$, then the solution of system (3.34) is

$$\begin{cases} x_3 = -ix_0, \\ x_2 = -ix_1, \\ x_1^2 = x_0x_4. \end{cases} \tag{3.36}$$

Systems (3.35) and (3.36) determine two conics C and $\overline{C}$ that are intersections of two complex conjugate 2-planes π and $\overline{\pi}$ defined by the first two equations of (3.35) and (3.36), respectively. The third (common) equation of these systems defines a real hypercone with a one-dimensional vertex $E_2 \wedge E_3$ defined by the equations $x_0 = x_1 = x_4 = 0$. This common third equation of (3.35) and (3.36) can be written in the form

$$\frac{x_1}{x_0} = \frac{x_4}{x_1}.$$

Define complex parameters $\sigma = t + is$ and $\overline{\sigma} = t - is$ on the conics C and $\overline{C}$ by setting on C

$$x_1 = \sigma\, x_0.$$

Then on this conic we obtain

$$x_2 = i\sigma\, x_0, \quad x_3 = ix_0, \quad x_4 = \sigma^2 x_0.$$

Similarly, on the conic $\overline{C}$, we obtain

$$\overline{x}_1 = \overline{\sigma}\, \overline{x}_0, \quad \overline{x}_2 = -i\overline{\sigma}\, \overline{x}_0, \quad \overline{x}_3 = -i\overline{x}_0, \quad \overline{x}_4 = \overline{\sigma}^2\, \overline{x}_0.$$

Taking in these relations $x_0 = 1$ (i.e., in nonhomogeneous coordinates), we find that

$$x = (1, \sigma, i\sigma, i, \sigma^2) \in C \tag{3.37}$$

and

$$\overline{x} = (1, \overline{\sigma}, -i\overline{\sigma}, -i, \overline{\sigma}^2) \in \overline{C}. \tag{3.38}$$

Consider the real straight line L defined by the points x and $\overline{x}$. Its arbitrary real point u has the form

$$u = \lambda x + \overline{\lambda}\overline{x}.$$

By (3.37) and (3.38), the coordinates u_0, u_1, u_2, u_3, and u_4 of this point are

$$\begin{cases} u_0 = \lambda + \overline{\lambda}, \\ u_1 = \lambda\sigma + \overline{\lambda}\overline{\sigma}, \\ u_2 = i(\lambda\sigma - \overline{\lambda}\overline{\sigma}), \\ u_3 = i(\lambda - \overline{\lambda}), \\ u_4 = \lambda\sigma^2 + \overline{\lambda}\overline{\sigma}^2. \end{cases}$$

Substituting these coordinates into equation (3.32), we can see that this equation becomes an identity. Thus, the straight line L belongs to the hypercubic (3.32), and *this hypersurface is ruled.*

Moreover, because the points x and $\overline{x}$ of the straight line L describe the conics C and $\overline{C}$, the hypercubic (3.32) is a join with directrices C and $\overline{C}$ (see Example 2.6). As a result, the hypercubic (3.32) is a real hypersurface X^3 with a degenerate Gauss map of rank two in the space $\mathbb{P}^4$. A tangent hyperplane to X^3 is determined by the points $x \in C$ and $\overline{x} \in \overline{C}$ and the straight lines tangent to C and $\overline{C}$ at these points (see Figure 3.2).

As we noted earlier, the common third equation of systems (3.35) and (3.36) determines a real second-order hypercone with a one-dimensional vertex $x_0 = x_1 = x_4 = 0$, and the 2-planes π and $\overline{\pi}$ do not pass through this vertex. These 2-planes intersect this hypercone along two conics C and $\overline{C}$. Complex conjugate points $x \in C$ and $\overline{x} \in \overline{C}$ define real rectilinear generators $L = x \wedge \overline{x}$ of the hypercubic X defined by (3.32). The tangent hyperplanes to X along L are constant because they are determined by the generator $x \wedge \overline{x}$ and the complex conjugate tangents to C and $\overline{C}$ at the points x and $\overline{x}$. Thus, we have a complete description of the hypercubic X defined by equation (3.32).

In this example $l = 1$ and $m = 1$. The hypercubic X^3 foliates into two families of complex conjugate hypercones.

Let us find the form of the matrices B^4, C_0, and C_1 for this hypercubic.

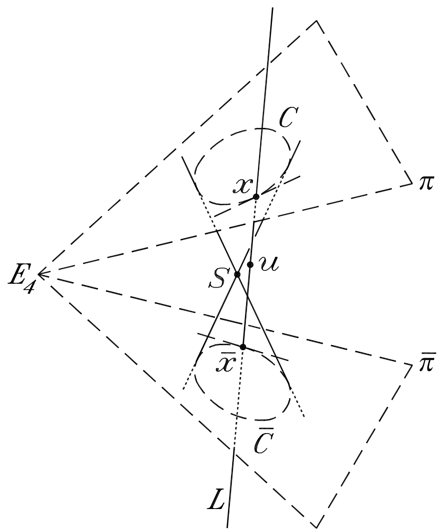


Figure 3.2

To this end, we choose as the vertices A_0 and A_1 of our moving frame the following real points:

$$A_0 = \frac{1}{2}(x + \overline{x}), \quad A_1 = \frac{1}{2i}(x - \overline{x}).$$

Applying (3.37) and (3.38), we find the coordinates of these points:

$$\begin{cases} A_0 = (1, t, -s, 0, t^2 - s^2), \\ A_1 = (0, s, t, 1, 2ts). \end{cases}$$

Differentiating the points A_0 and A_1, we find that

$$\begin{cases} dA_0 & = (0, dt, -ds, 0, 2(tdt - sds)) \\ & = (0, 1, 0, 0, 2t)dt + (0, 0, -1, 0, -2s)ds, \\ dA_1 & = (0, ds, dt, 0, 2(sdt + tds)) \\ & = (0, 0, 1, 0, 2s)dt + (0, 1, 0, 0, 2t)ds. \end{cases}$$

We take the points $(0, 1, 0, 0, 2t)$ and $(0, 0, -1, 0, -2s)$ as the vertices A_2 and A_3 of our moving frame:

$$A_2 = (0, 1, 0, 0, 2t), \quad A_3 = (0, 0, -1, 0, -2s).$$

Then the last equations take the form

$$\begin{cases} dA_0 = dt\, A_2 + ds\, A_3, \\ dA_1 = ds\, A_2 - dt\, A_3. \end{cases}$$

This shows that the points $A_1, A_1, A_2,$ and A_4 define the 3-plane T_L that is tangent to X along the rectilinear generator L.

Differentiating A_2 and A_3, we obtain

$$dA_2 = (0, 0, 0, 0, 2)dt, \quad dA_3 = (0, 0, 0, 0, -2)ds.$$

We take the point $(0, 0, 0, 0, 2)$ as the vertex A_4 of our moving frame:

$$A_4 = (0, 0, 0, 0, 2).$$

Then the differentials dA_2 and dA_3 take the form

$$\begin{cases} dA_2 = dt\, A_4, \\ dA_3 = -ds\, A_4. \end{cases}$$

Thus, for the frame $\{A_0, A_1, A_2, A_3, A_4\}$, we have the following equations of infinitesimal displacement:

$$\begin{cases} dA_0 = dt\, A_2 + ds\, A_3, \\ dA_1 = ds\, A_2 - dt\, A_3, \\ dA_2 = dt\, A_4, \\ dA_3 = -ds\, A_4, \\ dA_4 = 0. \end{cases}$$

Comparing these equations with (3.5), we see that

$$\begin{cases} \omega_0^2 = dt, & \omega_0^3 = ds, \\ \omega_1^2 = ds & \omega_1^3 = -dt, \\ \omega_2^4 = dt, & \omega_3^4 = -ds, \\ \omega_4^0 = \omega_4^1 = \omega_4^2 = \omega_4^3 = \omega_4^4 = 0. \end{cases}$$

Comparing the equations of the first two rows with (3.11) and the equations of the third row with (3.4), we find that

$$\begin{cases} c_{02}^2 = 1, & c_{03}^2 = 0, & c_{02}^3 = 0, & c_{03}^3 = 1, \\ c_{12}^2 = 0, & c_{13}^2 = 1, & c_{12}^3 = -1, & c_{13}^3 = 0, \\ b_{22}^4 = 1, & b_{23}^4 = 0, & b_{32}^4 = 0, & b_{33}^4 = -1. \end{cases}$$

From the definition of the matrices C_i and B^α, it follows that

$$C_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Equation (3.21) of the focus hypersurface F_L has the form

$$\begin{vmatrix} x_0 & -x_1 \\ x_1 & x_0 \end{vmatrix} = 0.$$

It follows that $x_1^2 = -x_0^2$, i.e., $x_1 = \pm i x_0$. Taking $x_0 = 1$, we get $x_1 = \pm i$. So, the focus hypersurface F_L in the generator $L = A_0 \wedge A_1$ consists of two complex conjugate points:

$$A_0 + iA_1 = x \text{ and } A_0 - iA_1 = \overline{x}.$$

Therefore, the hypersurface with a degenerate Gauss map defined in the space $\mathbb{P}^4$ by equation (3.40) does not have real singular points.

Examples 3.6 and 3.7 show how a parallel study of varieties in the real and complex spaces allows to understand their structure deeper.

3.4 The Sacksteder–Bourgain Hypersurface

3.4.1 The Sacksteder Hypersurface. In Cartesian coordinates (y^1, y^2, y^3, y^4), Sacksteder's hypersurface $S \subset \mathbb{A}^4$ is defined by the equation

$$y^4 = y^1 \cos(y^3) + y^2 \sin(y^3) \tag{3.39}$$

(see Sacksteder [S 60]).

We introduce in the space $\mathbb{A}^4$ homogeneous coordinates $(x^0, x^1, x^2, x^3, x^4)$ such that $y^\alpha = \frac{x^\alpha}{x^0}$, $\alpha = 1, 2, 3, 4$, and we enlarge this space to a projective space $\mathbb{P}^4$ by means of the improper hyperplane $\mathbb{P}^3_\infty$ defined by the equation $x^0 = 0$. Consider the natural extension of the hypersurface S in the space $\mathbb{P}^4$ and denote it by the same letter S. The equations for the hypersurface S can be represented in the parametric form

$$\left\{\begin{array}{l} x^0 = s, \\ x^1 = -sv\sin u + t\cos u, \\ x^2 = sv\cos u + t\sin u, \\ x^3 = su, \\ x^4 = t. \end{array}\right. \tag{3.40}$$

Equations (3.40) can be written in the form

$$X = sA_0 + tA_1,$$

where

$$\left\{\begin{array}{l} A_0 = (1, -v\sin u, v\cos u, u, 0), \\ A_1 = (0, \cos u, \sin u, 0, 1) \end{array}\right.$$

are points of the space $\mathbb{P}^4 = \mathbb{A}^4 \cup \mathbb{P}^3_\infty$. The straight lines $L = A_0 \wedge A_1$ are the generators of the hypersurface S defined by (3.39), because equation (3.39) is satisfied identically if we substitute the coordinates of the point X into this equation. Differentiating the points A_0 and A_1, we obtain

$$\left\{\begin{array}{l} dA_0 = A_2\, du + A_3 dv, \\ dA_1 = \qquad\qquad A_3 du, \end{array}\right.$$

where

$$\left\{\begin{array}{l} A_2 = (0, -v\cos u, -v\sin u, 1, 0), \\ A_3 = (0, -\sin u, \cos u, 0, 0). \end{array}\right.$$

It can be easily verified that the points A_0, A_1, A_2, and A_3 are linearly independent. Because u and v are constant along L, the tangent hyperplane

$T_L = A_0 \wedge A_1 \wedge A_2 \wedge A_3$ remains constant along the straight line $L = A_0 \wedge A_1$. This hyperplane, like a rectilinear generator L of the hypersurface S, depends solely on the parameters u and v. Thus, $\operatorname{rank} S = 2$.

We find the singular points (foci) of a generator $L = A_0 \wedge A_1$ of the hypersurface $S \subset \mathbb{P}^4$ in the same manner as for the general case in Section 2.3. A point $X = sA_0 + tA_1$ is the focus of this generator if $dX \in A_0 \wedge A_1$, whence it follows that, for the focus,

$$s(A_2 du + A_3 dv) + tA_3 du = 0.$$

Because the points A_2 and A_3 are linearly independent, it follows that

$$\begin{cases} s\,du \qquad\qquad = 0, \\ t\,du + s\,dv = 0. \end{cases} \tag{3.41}$$

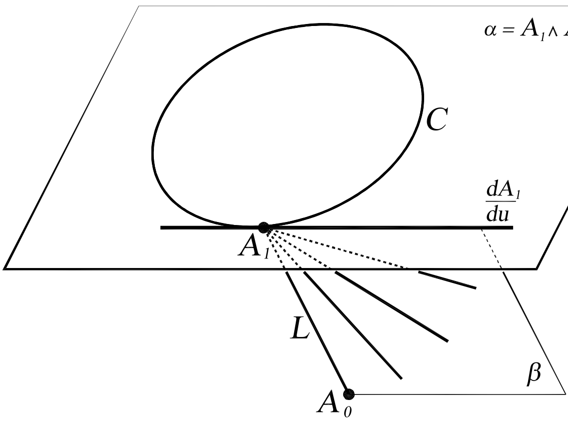


Figure 3.3

This system should have a nontrivial solution relative to du and dv, which defines a focal direction on S. Consequently,

$$\det \begin{pmatrix} s & 0 \\ t & s \end{pmatrix} = 0$$

and $s^2 = 0$. This means that the point A_1 (the point at infinity of a rectilinear generator of the hypersurface S) is the double focus of the line $A_0 \wedge A_1$. By equations (3.41), the torses on the hypersurface S are defined by the equation $du = 0$, and these torses are the pencils of straight lines with centers at the points $A_1 = \{0, \cos u, \sin u, 0, 1\}$ located in the 2-planes $\beta = A_0 \wedge A_1 \wedge A_3$, where $A_3 = \frac{dA_1}{du}$. With respect to a fixed frame, the planes β are defined by the equations

$$\left\{ \begin{array}{l} x^3 = ux^0, \\ x^4 = x^1 \cos u + x^2 \sin u. \end{array} \right. \tag{3.42}$$

The 2-planes β belong to S and are its plane generators (see Figure 3.3). They form a one-parameter family depending on the parameter u. But $T_x(S)$ is fixed only along a generator $L = A_0 \wedge A_1 \subset \beta$. This is why $\operatorname{rank} X = 2$ (not 1). The point A_1 describes the conic C in $\mathbb{P}^3_\infty$ defined by the equations

$$\left\{ \begin{array}{l} x^0 = 0, \\ x^3 = 0, \\ (x^1)^2 + (x^2)^2 = (x^4)^2, \end{array} \right. \tag{3.43}$$

which can be easily derived from $A_1 = (0, \cos u, \sin u, 0, 1)$. Because $A_3 = \frac{dA_1}{du}$, the straight line $A_1 \wedge A_3$ is tangent to the conic C at the point A_1.

Thus, the hypersurface S defined by equation (3.39) in the space $\mathbb{A}^4$, has no singularities in the proper domain of this space, because they have "retreated" to the hyperplane at infinity $\mathbb{P}^3_\infty$ of this space. On the other hand, the hypersurface S is not a cylinder. Such hypersurfaces are called *twisted cylinders.*

The example we discussed can be easily generalized. Let γ be an arbitrary complete smooth curve in the hyperplane at infinity H_∞ of an affine space $\mathbb{A}^N$. Suppose that this curve is described by the point $A_1 = A_1(u)$. We set $A_3 = \frac{dA_1}{du}$, and let $\beta = \beta(u)$ be the smooth family of proper tangent 2-planes of the curve γ. These 2-planes form a complete regular variety $X = V^3_2$ of rank $r = 2$ on which the Monge–Ampère foliation is formed by the pencils of straight lines with centers at the points A_1 located in the 2-planes β. The proof of this assertion differs little from our investigation of the structure of hypersurface (3.39) in $\mathbb{A}^4$.

3.4.2 The Bourgain Hypersurface. Recently Wu [Wu 95] published an example of a noncylindrical algebraic hypersurface with a degenerate Gauss map in a Euclidean space $\mathbb{E}^4$ that has a degenerate Gauss mapping but does not have singularities. This example was constructed (but not published) by Bourgain (see also Ishikawa [I 98, 99a, 99b]).

This example can be considered in an affine space $\mathbb{A}^4$ (and even in a projective space $\mathbb{P}^4$). In the affine space $\mathbb{A}^4$, in Cartesian coordinates (x_1, x_2, x_3, x_4)

the equation of the Bourgain hypersurface B is

$$x_1 x_4^2 + x_2(x_4 - 1) + x_3(x_4 - 2) = 0 \tag{3.44}$$

(see Wu [Wu 95] or Ishikawa [I 98, 99a, 99b]). Equation (3.44) can be written in the form

$$x_1 x_4^2 + (x_2 + x_3)x_4 - (x_2 + 2x_3) = 0. \tag{3.45}$$

In (3.45) make the following admissible change of Cartesian coordinates:

$$x_2 + x_3 \to x_2, \quad x_2 + 2x_3 \to x_3.$$

Then equation (3.45) becomes

$$x_1 x_4^2 + x_2 x_4 - x_3 = 0. \tag{3.46}$$

Introduce homogeneous coordinates in $\mathbb{A}^4$ by setting $x_i = \dfrac{z_i}{z_0}$, $i = 1, 2, 3, 4$. Then equation (3.46) takes the form

$$g(z_0, z_1, z_2, z_3, z_4) = z_1 z_4^2 + z_0 z_2 z_4 - z_0^2 z_3 = 0. \tag{3.47}$$

Equation (3.47) defines a cubic hypersurface G in the space $\overline{\mathbb{A}}^4 = \mathbb{A}^4 \cup \mathbb{P}^3_\infty$ which is an enlarged space $\mathbb{A}^4$, i.e., it is the space $\mathbb{A}^4$ enlarged by the hyperplane at infinity $\mathbb{P}^3_\infty$ (whose equation is $z_0 = 0$).

Denote by $E_\alpha,\ \alpha = 0, 1, 2, 3, 4$, fixed basis points of the space $\overline{\mathbb{A}}^4$. Suppose that these points have constant normalizations, i.e., that $dE_\alpha = 0$. An arbitrary point $z \in \overline{\mathbb{A}}^4$ can be written in the form $z = \sum_\alpha z_\alpha E_\alpha$. We take a proper point of the space $\overline{\mathbb{A}}^4$ as the point E_0, and take four linearly independent points at infinity as the points E_1, E_2, E_3, E_4.

Equation (3.47) shows that the proper straight line $E_0 \wedge E_4$ defined by the equations $z_1 = z_2 = z_3 = 0$ and the plane at infinity defined by the equations $z_0 = z_4 = 0$ belong to the hypercubic G defined by equation (3.47).

We write the equations of the hypercubic G in a parametric form. To this end, we set

$$z_0 = 1, \quad z_4 = p, \quad z_1 = u, \quad z_3 = pv.$$

Then it follows from (3.47) that

$$z_2 = v - pu.$$

This implies that an arbitrary point $z \in G$ can be written as

$$z = E_0 + uE_1 + vE_2 + p(E_4 - uE_2 + vE_3). \tag{3.48}$$

The parameters p, u, and v are nonhomogeneous coordinates on the hypercubic G.

Let us find singular points of the hypercubic G. Such points are defined by the equations $\dfrac{\partial g}{\partial z_\alpha} = 0$. It follows from (3.47) that

$$\begin{cases} \dfrac{\partial g}{\partial z_0} = z_2 z_4 - 2 z_0 z_3, \\ \dfrac{\partial g}{\partial z_1} = z_4^2, \quad \dfrac{\partial g}{\partial z_2} = z_0 z_4, \quad \dfrac{\partial g}{\partial z_3} = -z_0^2, \\ \dfrac{\partial g}{\partial z_4} = 2 z_1 z_4 + z_0 z_2. \end{cases} \tag{3.49}$$

All these derivatives vanish simultaneously if and only if $z_0 = z_4 = 0$. Thus the 2-plane at infinity $\sigma = E_1 \wedge E_2 \wedge E_3$ is the locus of singular points of the hypercubic G.

Consider a point $A_0 = E_0 + pE_4 = (1, 0, 0, 0, p)$ on the straight line $E_0 \wedge E_4$. It follows from (3.47) that to the point A_0 there is the corresponding straight line $a(p)$ in the 2-plane at infinity σ, and the equation of this straight line is

$$p^2 z_1 + p z_2 - z_3 = 0. \tag{3.50}$$

The family of straight lines $a(p)$ depends on the parameter p. Its envelope is determined by equation (3.50) and the equation

$$2p z_1 + z_2 = 0.$$

Excluding parameter p from the last two equations, we find that the envelope is the conic C defined by the equation

$$z_2^2 + 4 z_1 z_3 = 0. \tag{3.51}$$

The straight line $a(p)$ is tangent to the conic C at the point

$$A_1(p) = E_1 - 2pE_2 - p^2 E_3. \tag{3.52}$$

Equation (3.52) is a parametric equation of the conic C. The point

$$\frac{dA_1}{dp} = -2(E_2 + pE_3) \tag{3.53}$$

belongs to the tangent line to the conic C at the point $A_1(p)$.

Consider the 2-planes $\tau = A_0 \wedge A_1 \wedge \frac{dA_1}{dp}$. Such 2-planes are completely determined by the location of the point A_0 on the straight line $E_0 \wedge E_4$, and they form a one-parameter family. All these 2-planes belong to the hypercubic G. In fact, represent an arbitrary point z of the 2-plane τ in the form

$$\begin{aligned} z &= \alpha A_0 + \beta A_1 - \frac{1}{2}\gamma \frac{dA_1}{dp} \\ &= \alpha E_0 + \beta E_1 + (-2p\beta + \gamma)E_2 + (-p^2\beta + p\gamma)E_3 + p\alpha E_4. \end{aligned} \tag{3.54}$$

The coordinates of the point z are

$$z_0 = \alpha, \quad z_1 = \beta, \quad z_2 = \gamma - 2p\beta, \quad z_3 = p(\gamma - p\beta), \quad z_4 = p\alpha. \tag{3.55}$$

Substituting these values of the coordinates into equation (3.47), one can see that equation (3.47) is identically satisfied. Thus the hypercubic G is foliated into a one-parameter family of 2-planes $\tau(p) = A_0 \wedge A_1 \wedge \frac{dA_1}{dp}$.

In a 2-plane $\tau(p)$ consider a pencil of straight lines with center A_1. The straight lines of this pencil are defined by the point A_1 and the point $A_2 = E_2 + pE_3 + q(E_0 + pE_4)$. The straight lines $A_1 \wedge A_2$ depend on two parameters p and q. These lines belong to the 2-plane $\tau(p)$, and along with this 2-plane they belong to the hypercubic G. Thus they form a foliation on the hypercubic G.

We prove that this foliation is a Monge-Ampère foliation. In the space $\overline{\mathbb{A}}^4$, we introduce the moving frame formed by the points

$$\begin{cases} A_0 = E_0 + pE_4, \\ A_1 = E_1 - 2pE_2 - p^2E_3, \\ A_2 = E_2 + pE_3 + qE_0 + pqE_4, \\ A_3 = E_3, \\ A_4 = E_4. \end{cases} \tag{3.56}$$

It is easy to prove that these points are linearly independent, and the points E_α can be expressed in terms of the points A_α as follows:

$$\begin{cases} E_0 = A_0 - pA_4, \\ E_1 = A_1 + 2pA_2 - p^2A_3 - 2pqA_0, \\ E_2 = A_2 - pA_3 - qA_0, \\ E_3 = A_3, \\ E_4 = A_4. \end{cases} \tag{3.57}$$

Consider a displacement of the straight lines $A_1 \wedge A_2$ along the hypercubic G. Suppose that Z is an arbitrary point of this straight line,

$$Z = A_1 + \lambda A_2. \tag{3.58}$$

Differentiating (3.58) and taking into account (3.57) and $dE_\alpha = 0$, we find that

$$dZ \equiv (2qdp + \lambda dq)A_0 + \lambda dp(A_3 + qA_4) \pmod{A_1, A_2}. \tag{3.59}$$

It follows from relation (3.59) that:

1. A tangent hyperplane to the hypercubic G is spanned by the points A_1, A_2, A_0 and $A_3 + qA_4$. This hyperplane is fixed when the point Z moves along the straight line $A_1 \wedge A_2$. Thus, G is a hypersurface with a degenerate Gauss map of rank 2, and the straight lines $A_1 \wedge A_2$ form a Monge-Ampère foliation on G.

2. The system of equations

$$\begin{cases} 2q\,dp\ + \lambda dq = 0, \\ \lambda\, dp \qquad\qquad = 0 \end{cases} \tag{3.60}$$

   defines singular points on the straight line $A_1 \wedge A_2$, and on the hypersurface G it defines torses. The system of equations (3.60) has a nontrivial solution with respect to dp and dq if and only if its determinant vanishes: $\lambda^2 = 0$. Hence by (3.58), a singular point on the straight line $A_1 \wedge A_2$ coincides with the point A_1. For $\lambda = 0$, system (3.60) implies that $dp = 0$, i.e., $p =$ const. Thus it follows from (3.52) that the point $A_1 \in C$ is fixed, and as a result, the torse corresponding to this constant parameter p is a pencil of straight lines with the center A_1 located in the 2-plane $\tau(p) = A_0 \wedge A_1 \wedge A_2$.

3. All singular points of the hypercubic G belong to the conic $C \subset \mathbb{P}^\infty$ defined by equation (3.52). Thus if we consider the hypercubic G in an affine space $\mathbb{A}^4$, then on G there are no singular points in a proper part of this space.

4. The hypercubic G considered in the proper part of an affine space is not a cylinder because its rectilinear generators do not belong to a bundle of parallel straight lines. A two-parameter family of rectilinear generators of G decomposes into a one-parameter family of plane pencils of parallel lines.

None of these properties characterizes Bourgain's hypersurfaces completely: they are necessary but not sufficient for these hypersurfaces. The following theorem gives a necessary and sufficient condition for a hypersurface to be of Bourgain's type.

Theorem 3.8. *Let l be a proper straight line of an affine space $\mathbb{A}^4$ enlarged by the hyperplane at infinity $\mathbb{P}^3_\infty$, and let C be a conic in the 2-plane $\sigma \subset \mathbb{P}^3_\infty$. Suppose that the straight line l and the conic C are in a projective correspondence. Let $A_0(p)$ and $A_1(p)$ be two corresponding points of l and C, and let τ be the 2-plane passing through the point A_0 and tangent to the conic C at the point A_1. Then*

(a) *when the point A_0 is moving along the straight line l, the plane τ describes a Bourgain hypersurface, and*

(b) *any Bourgain hypersurface satisfies the described construction.*

Proof. Necessity (b) of the theorem hypotheses follows from our previous considerations. We prove sufficiency (a) of these hypotheses. Take a fixed frame $\{E_\alpha\}$, $\alpha = 0, 1, 2, 3, 4$, in the space $\mathbb{A}^4$ enlarged by the plane at infinity $\mathbb{P}^3_\infty$ as follows: its point E_0 belongs to l, the point E_4 is the point at infinity of l, and the points E_1, E_2, and E_3 are located at the 2-plane at infinity σ in such a way that a parametric equation of the straight line l is $A_0 = E_0 + pE_4$, and the equation of C has the form (3.52). The plane τ is defined by the points A_0, A_1, and $\frac{dA_1}{dp}$. The parametric equations of this plane have the form (3.55). Excluding the parameters α, β, γ, and p from these equations, we return to the cubic equation (3.47) defining the Bourgain hypersurface B in homogeneous coordinates. □

The method of construction of the Bourgain hypersurface used in the proof of Theorem 3.8 goes back to the classical methods of projective geometry developed by Steiner [St 32] and Reye [R 68].

3.4.3 Local Equivalence of Sacksteder's and Bourgain's Hypersurfaces. In Section 3.4.2, we investigated Bourgain's hypersurface B. In particular, we proved that, as was the case for the Sacksteder hypersurface S, the Bourgain hypersurface has no singularities because they "go to infinity" and compose a conic C in the hyperplane at infinity H_∞. This analysis suggests an idea that Bourgain's and Sacksteder's hypersurfaces should be equivalent. Moreover, this analysis showed that a hypersurface constructed in these examples is torsal, i.e., it is stratified into a one-parameter family of plane pencils of straight lines.

Now we prove the following theorem.

Theorem 3.9. *The Sacksteder hypersurface S and the Bourgain hypersurface B are locally equivalent, and the former is the standard covering of the latter.*

Proof. In a Euclidean space $\mathbb{E}^4$, in Cartesian coordinates (x_1, x_2, x_3, x_4), the equation of the Sacksteder hypersurface S (cf. equation (3.39)) has the form

$$x_4 = x_1 \cos x_3 + x_2 \sin x_3. \tag{3.61}$$

The right-hand side of this equation is a function on the manifold $M^3 = \mathbb{R}^2 \times \mathbb{S}^1$ because the variable x_3 is cyclic. Equation (3.61) defines a hypersurface on the manifold $M^3 \times \mathbb{R}$. The circumference $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$, where $\mathbb{Z}$ is the set of integers, has a natural projective structure of $\mathbb{P}^1$. In the homogeneous coordinates $x_3 = \frac{u}{v}$, the mapping $\mathbb{S}^1 \to \mathbb{P}^1$, can be written as $x_3 \to (u, v)$. By removing the point $\{v = 0\}$ from $\mathbb{S}^1$, we obtain a one-to-one correspondence

$$\mathbb{S}^1 \backslash \{v = 0\} \longleftrightarrow \mathbb{R}^1. \tag{3.62}$$

Now we can consider the Sacksteder hypersurface S in $\mathbb{A}^4$ or, if we enlarge $\mathbb{A}^4$ by the plane at infinity $\mathbb{P}^3_\infty$, in the space $\mathbb{P}^4$.

Next we show how by applying the mapping $\mathbb{S}^1 \to \mathbb{P}^1$, we can transform equation (3.61) of the Sacksteder hypersurface S into equation (3.47) of the Bourgain hypersurface B. We write this mapping in the form

$$x_3 = 2\arctan\frac{u}{v}, \quad \frac{u}{v} \in \mathbb{R}, \ |x_3| < \pi. \tag{3.63}$$

It follows from (3.63) that

$$\begin{cases} \dfrac{u}{v} = \tan\dfrac{x_3}{2}, \\ \cos x_3 = \dfrac{1 - \tan^2\dfrac{x_3}{2}}{1 + \tan^2\dfrac{x_3}{2}} = \dfrac{v^2 - u^2}{v^2 + u^2}, \\ \sin x_3 = \dfrac{2\tan\dfrac{x_3}{2}}{1 + \tan^2\dfrac{x_3}{2}} = \dfrac{2uv}{v^2 + u^2}. \end{cases} \tag{3.64}$$

Substituting these expressions into equation (3.61), we find that

$$x_4(u^2 + v^2) = x_1(v^2 - u^2) + 2x_2uv,$$

i.e.,

$$(x_4 + x_1)u^2 + (x_4 - x_1)v^2 - 2x_2uv = 0. \tag{3.65}$$

Make a change of variables

$$z_1 = x_4 - x_1, \quad z_2 = -2x_2, \quad z_3 = x_1 + x_4, \quad z_0 = u, \quad z_4 = v.$$

As a result, we reduce equation (3.65) to equation (3.47). It follows that the Sacksteder hypersurface S defined by equation (3.44) is locally equivalent to the Bourgain hypersurface defined by equation (3.47).

Note also that if the cyclic parameter x_3 changes on the entire real axis $\mathbb{R}$, then we obtain the standard covering of the Bourgain hypersurface B by means of the Sacksteder hypersurface S. $\square$

3.4.4 Computation of the Matrices C_i and B^α for Sacksteder–Bourgain Hypersurfaces. We now compute the matrices C_i and B^α for Sacksteder–Bourgain's hypersurfaces. In Section 3.4.2, for Sacksteder–Bourgain's hypersurface defined by parametric equation (3.39), we choose the following vertices of the moving frame $\{A_0, A_1, A_2, A_3, A_4\}$:

$$\left\{\begin{array}{l} A_0 = (1, -v\sin u, v\cos u, u, 0), \\ A_1 = (0, \cos u, \sin u, 0, 1), \\ A_2 = (0, -v\cos u, -v\sin u, 1, 0), \\ A_3 = (0, -\sin u, \cos u, 0, 0). \end{array}\right. \tag{3.66}$$

Differentiating these points, we find that

$$\left\{\begin{array}{ll} dA_0 = A_2 du & + A_3 dv, \\ dA_1 = & \quad A_3 du, \end{array}\right. \tag{3.67}$$

and

$$\left\{\begin{array}{l} dA_2 = (0, -\cos u dv + v\sin u du, -\sin u dv - v\cos u du, 1, 0) \\ \qquad = (0, \sin u, -\cos u, 0, 0) v du + (0, -\cos u, -\sin u, 0, 0) dv, \\ dA_3 = (0, -\cos u, -\sin u, 0, 0) du. \end{array}\right.$$

We take as the point A_4 of our moving frame the point

$$A_4 = (0, -\cos u, -\sin u, 0, 0).$$

Then the differentials dA_2 and dA_3 take the form:

$$\left\{\begin{array}{ll} dA_2 = -v\, du A_3 & + dv A_4, \\ dA_3 = & \quad du A_4. \end{array}\right. \tag{3.68}$$

It follows from equations (3.67) and (3.68) that

$$\left\{\begin{array}{ll} \omega_0^2 = du, & \omega_0^3 = dv, \\ \omega_1^2 = 0, & \omega_1^3 = \omega_0^2, \\ \omega_2^4 = \omega_0^3, & \omega_3^4 = \omega_0^2. \end{array}\right. \tag{3.69}$$

Comparing equations (3.69) with (3.11) and (3.4), we find the following form of the matrices C_i and B^α:

$$C_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B^4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.70}$$

3.5 Complete Varieties with Degenerate Gauss Maps in Real Projective and Non-Euclidean Spaces

3.5.1 Parabolic Varieties. With varieties with degenerate Gauss maps in a projective space $\mathbb{P}^N$ there are associated the so-called *parabolic varieties* in simply connected Riemannian spaces of constant curvature.

In Section 2.3 we defined the index $l(x)$ of relative nullity (the Gauss defect) of the variety X, $\dim X = n$, at the point x. It was related to the rank r of X by the equation $l = n - r$. If $l(x) > 0$, then the point $x \in X$ is called a *parabolic* point of the variety X. If all points of a variety X are parabolic, then the variety X is called *parabolic.*

Varieties X of a Riemannian space $\mathbb{V}^N$ of a constant Gauss defect $l(x) = l$ are called *l-parabolic varieties* (cf. the papers [Bor 82, 85] by Borisenko).

We will now study complete l-parabolic varieties in real simply connected Riemannian spaces $\mathbb{V}^N_c$ of constant curvature c. If $c = 0$, then $\mathbb{V}^N_c$ is the Euclidean space $\mathbb{E}^N$. If $c > 0$, then $\mathbb{V}^N_c$ is the elliptic space $\mathbb{S}^N$. If $c < 0$, then $\mathbb{V}^N_c$ is the hyperbolic space $\mathbb{H}^N$. Each of these spaces admits a geodesic mapping into the space $\mathbb{P}^N$, which is usually called the *projective realization* of the corresponding space $\mathbb{V}^N_c$.

The Euclidean space $\mathbb{E}^N$ is realized in the projective space $\mathbb{P}^N$ from which a hyperplane $\mathbb{E}_\infty$ has been removed (this hyperplane is called *improper* or *the hyperplane at infinity*), and the proper domain of the space $\mathbb{E}^N$ can be identified with the open simply connected manifold $\mathbb{P}^N \backslash \mathbb{E}_\infty$. The elliptic space $\mathbb{S}^N$ is realized in the entire projective space $\mathbb{P}^N$, because the absolute of $\mathbb{S}^N$ is an imaginary hyperquadric and its proper domain coincides with the entire space $\mathbb{P}^N$. Finally, the hyperbolic space $\mathbb{H}^N$ is realized in the part of the projective space $\mathbb{P}^N$ lying within the convex hyperquadric that is the absolute of this space. This open simply connected domain is the proper domain of the hyperbolic space $\mathbb{H}^N$. We denote by G the proper domain of the simply connected space $\mathbb{V}^N_c$ in all these cases.

Let X, $\dim X = n$, be a complete parabolic variety of a space $\mathbb{V}^N_c$ of constant curvature. Suppose that X has a constant Gauss defect l. Let $\overline{X}$ be the image of X in the domain G of the space $\mathbb{P}^N$ in which the space $\mathbb{V}^N_c$ is realized, and let $\widetilde{X}$ be the natural extension of this image in the space $\mathbb{P}^N$, so that $\overline{X} = \widetilde{X} \cap G$. In this extension, l-dimensional plane generators of the variety $\overline{X}$ are complemented by improper elements from the complement $\mathbb{P}^N \backslash G$. Because the variety X has a constant Gauss defect l, its realization $\widetilde{X}$ in the space $\mathbb{P}^N$ is a variety with a degenerate Gauss map of rank $r = n - l < n$. The variety $\widetilde{X}$ bears l-dimensional plane generators L, and each L carries the focus hypersurface F_L of degree r, which is the set of all singularities of L. Thus a variety X is regular if and only if the real part Re F_L of its focus hypersurfaces is located *outside* of the proper domain G of the space $\mathbb{V}_c^N$.

One of the important problems of multidimensional differential geometry is the finding of complete l-parabolic varieties X without singularities in spaces $\mathbb{V}_c^N$ of constant curvature. Theorem 3.2 implies the following result.

Theorem 3.10. *Let V^m be a complete l-parabolic variety of a simply connected space $\mathbb{V}_c^N$ of constant curvature. Let $\overline{X} = f(X)$ be the image of X in the proper domain G of the space $\mathbb{P}^N$ in which the space $\mathbb{V}_c^N$ is realized, and let $\widetilde{X}$ be the natural extension of this image in the space $\mathbb{P}^N$. The variety X is regular if and only if the real parts* Re F_L *of the focus hypersurfaces F_L belonging to generators L of the variety $\widetilde{X}$ lie outside of the proper domain $G \subset \mathbb{P}^N$.*

Let us examine the content of Theorem 3.10 for the different kinds of spaces $\mathbb{V}_c^N$ of constant curvature.

If $c = 0$, then $\mathbb{V}_c^N$ is the Euclidean space $\mathbb{E}^N$, and $\mathbb{P}^N \backslash \mathbb{E}_\infty$ is the proper domain of its projective realization. Thus a complete l-parabolic variety X of the space $\mathbb{E}^N$ is regular if and only if the real part Re F_L of the focus variety F_L of each plane generator L of the corresponding variety $\widetilde{X} \subset \mathbb{P}^N$ coincides with the intersection $L \cap \mathbb{E}_\infty$ and constitutes a ρ-fold $(l-1)$-plane, where $0 < \rho \leq r$.

If $c > 0$, then $\mathbb{V}_c^N$ is the elliptic space $\mathbb{S}^N$, and its proper domain coincides with the entire space $\mathbb{P}^N$. Thus a complete l-parabolic variety X of the space $\mathbb{S}^N$ is regular if and only if the focus hypersurface F_L of each plane generator L of the corresponding variety $\widetilde{X}$ is pure imaginary. This is possible only if the Gauss defect $\delta_\gamma = l$ of the variety X is odd.

If $c < 0$, then $\mathbb{V}_c^N$ is the hyperbolic space $\mathbb{H}^N$, and the proper domain of its realization lies inside the absolute of this space. Thus a complete l-parabolic variety X of the space $\mathbb{H}^N$ is regular if and only if the real part Re F_L of the focus hypersurface F_L of each plane generator L of the corresponding variety $\widetilde{X}$ lies outside of or on the absolute of this space.

Parabolic surfaces of a three-dimensional space $\mathbb{V}_c^3$ of constant curvature allow an especially simple description. In $\mathbb{P}^3$, with each parabolic surface V^2 there is associated a corresponding torse, each rectilinear generator of which possesses a focus point. The locus of these focus points constitutes an edge of regression of the surface V^2. If $c = 0$, then this edge of regression must belong to the improper plane $\mathbb{E}_\infty$, i.e., the edge of regression is a plane curve. But this is possible if and only if the edge of regression degenerates into a point. Therefore, a projective realization of a hyperbolic surface V^2 of a three-dimensional Euclidean space $\mathbb{E}^3$ is a cone with its vertex in the improper plane $\mathbb{E}_\infty$. Thus the surface V^2 itself is a cylinder. Hence in the space $\mathbb{E}^3$ there are no other regular parabolic surfaces except the cylinders.

If $c > 0$, i.e., if we have the elliptic space $\mathbb{S}^3$, then there are no regular parabolic surfaces, because the edge of regression of the torse V^2 is always real. Finally, if $c < 0$, i.e., if we have the hyperbolic space $\mathbb{H}^3$, then there are regular parabolic surfaces, because the real edge of regression of the torse V^2 can be located outside of the absolute.

Thus we have proved the following result.

Theorem 3.11. *In the Euclidean space* $\mathbb{E}^3$*, only cylinders are regular parabolic surfaces. In the space* $\mathbb{S}^3$*, there are no regular parabolic surfaces at all, and in the space* $\mathbb{H}^3$*, regular parabolic surfaces exist and depend on two arbitrary functions of one variable.*

The last statement follows from the fact that a torse in $\mathbb{P}^3$ is completely defined by its edge of regression, i.e., by an arbitrary space curve, but these curves are defined in $\mathbb{P}^3$ by two arbitrary functions of one variable, as indicated in the theorem. Of course, the functions of one variable mentioned in Theorem 3.11 are not completely arbitrary: they must satisfy some inequalities guaranteeing that the variety $\operatorname{Sing} X$ does not belong to the proper domain of the space $\mathbb{H}^3$ located inside of the absolute.

3.5.2 Examples. In order to construct examples of parabolic varieties without singularities in a simply connected space $\mathbb{V}_c^N$ of constant curvature c, we first find such examples in the real projective space $\mathbb{RP}^N$.

In the real projective space $\mathbb{RP}^N$, we have already considered in Section 3.4 an example of such varieties—the Sacksteder–Bourgain hypersurfaces in $\mathbb{A}^4$. Note that this kind of variety will be considered again in Section 5.2

Now we construct another example of a three-dimensional variety with a degenerate Gauss map of rank two in the real space $\mathbb{RP}^N$, $N \geq 4$, which does not have real singular points.

Example 3.12. We consider in $\mathbb{RP}^N$, $N \geq 4$, a three-dimensional variety $X = V_2^3$ of rank two with imaginary focus hypersurface F_L. Equations (2.5), (3.3), (3.4), and (3.11) defining this variety in $\mathbb{RP}^N$ take the form

$$\omega_0^\alpha = \omega_1^\alpha = 0, \quad \alpha = 4, \ldots, N, \tag{3.71}$$

$$\omega_1^p = c_q^p \omega_0^q, \quad \omega_p^\alpha = b_{pq}^\alpha \omega_0^q, \quad p, q = 2, 3, \tag{3.72}$$

while equation (3.21), defining the foci on the generator $A_0 \wedge A_1$ of this variety, is written as

$$\det \begin{pmatrix} x^0 + x^1 c_2^2 & x^1 c_3^2 \\ x^1 c_2^3 & x^0 + x^1 c_3^3 \end{pmatrix} = 0.$$

Setting $\frac{x^0}{x^1} = -\lambda$, we reduce this equation to the form

$$\lambda^2 - (c_2^2 + c_3^3)\lambda + (c_2^2 c_3^3 - c_3^2 c_2^3) = 0.$$

Because the focus hypersurface F_L is assumed to be imaginary, this equation has complex-conjugate roots $\lambda = c_2 \pm ic_3$, where $c_3 \neq 0$. As a result, a real transformation converts the matrix $C = (c^p_q)$ to the form

$$C = \begin{pmatrix} c_2 & c_3 \\ -c_3 & c_2 \end{pmatrix}.$$

Substituting these values for the components of the matrix C into equations (3.9), and taking into account that $c_3 \neq 0$, we find that

$$b^\alpha_{22} + b^\alpha_{33} = 0.$$

In view of this, the symmetric matrices B^α can be written in the form

$$B^\alpha = \begin{pmatrix} b^\alpha_2 & b^\alpha_3 \\ b^\alpha_3 & -b^\alpha_2 \end{pmatrix}.$$

Then equations (3.72) assume the form

$$\begin{cases} \omega^2_1 = \quad c_2\omega^2_0 + c_3\omega^3_0, \\ \omega^3_1 = -\ c_3\omega^2_0 + c_2\omega^3_0, \end{cases} \tag{3.73}$$

$$\begin{cases} \omega^\alpha_2 = b^\alpha_2\omega^2_0 + b^\alpha_3\omega^3_0, \\ \omega^\alpha_3 = b^\alpha_3\omega^2_0 - b^\alpha_2\omega^3_0. \end{cases} \tag{3.74}$$

We now find the osculating subspace T^2_x of our variety $X \subset \mathbb{RP}^N$. Its tangent subspace T_x is spanned by the points A_0, A_1, A_2, and A_3. Because by (3.74),

$$dA_2 \equiv (b^\alpha_2\omega^2_0 + b^\alpha_3\omega^3_0)A_\alpha \pmod{T_x},$$
$$dA_3 \equiv (b^\alpha_3\omega^2_0 - b^\alpha_2\omega^3_0)A_\alpha \pmod{T_x},$$

the subspace T^2_x comprises the linear span of the subspace T_x and the points $B_2 = b^\alpha_2 A_\alpha$ and $B_3 = b^\alpha_3 A_\alpha$.

Two cases are possible:

(a) The points B_2 and B_3 are linearly independent. Then $\dim T_x = 5$, and the dimension of the space $N \geq 5$.

(b) The points B_2 and B_3 are linearly dependent. Then $\dim T_x = 4$, and $N \geq 4$.

We examine these two cases in turn. In case (a), we specialize the moving frames in $\mathbb{RP}^N$ in such a fashion that $A_4 = B_2$ and $A_5 = B_3$. Then equations (3.74) take the form

$$\begin{aligned} &\omega_2^4 = \omega_0^2, \quad &\omega_2^5 = \omega_0^3, \quad &\omega_2^\lambda = 0, \\ &\omega_3^4 = -\omega_0^3, \quad &\omega_3^5 = \omega_0^2, \quad &\omega_3^\lambda = 0, \end{aligned} \tag{3.75}$$

where $\lambda = 6, \ldots, N$. Therefore the variety X in case (a) is determined by the system of Pfaffian equations (3.71), (3.73), and (3.75).

Next, we investigate the consistency of this system by means of the Cartan test (see Section 1.2.6). For this purpose we adjoin to Pfaffian equations (3.75) the exterior quadratic equations obtained as the result of exterior differentiation of these Pfaffian equations. Exterior differentiation of equations (3.71) leads to identities, by virtue of (3.73), and (3.75). Exterior differentiation of equations (3.73) yields

$$\begin{aligned} &(\Delta c_2 - c_3(\omega_2^3 + \omega_3^2)) \wedge \omega_0^2 + (\Delta c_3 + c_3(\omega_2^2 - \omega_3^3)) \wedge \omega_0^3 = 0, \\ -\, &(\Delta c_3 - c_3(\omega_2^2 - \omega_3^3)) \wedge \omega_0^2 + (\Delta c_2 + c_2(\omega_2^3 - \omega_3^2)) \wedge \omega_0^3 = 0, \end{aligned} \tag{3.76}$$

where

$$\begin{aligned} \Delta c_2 &= dc_2 + c_2(\omega_0^0 - \omega_1^1) - \omega_1^0 + ((c_2)^2 - (c_3)^2)\omega_0^1, \\ \Delta c_3 &= dc_3 + c_3(\omega_0^0 - \omega_1^1) + 2c_2c_3\omega_0^1. \end{aligned}$$

Exterior differentiation of equations (3.75) gives

$$\begin{aligned} &(\omega_0^0 + \omega_4^4 + c_2\omega_0^1 - 2\omega_2^2) \wedge \omega_0^2 + (\omega_2^3 - \omega_3^2 + \omega_5^4 + c_3\omega_0^1) \wedge \omega_0^3 = 0, \\ &(\omega_2^3 - \omega_3^2 + \omega_5^4 + c_3\omega_0^1) \wedge \omega_0^2 - (\omega_0^0 + \omega_4^4 + c_2\omega_0^1 - 2\omega_2^2) \wedge \omega_0^3 = 0, \\ &(\omega_4^5 - c_3\omega_0^1 - 2\omega_2^3) \wedge \omega_0^2 + (\omega_0^0 - \omega_2^2 - \omega_3^3 + \omega_5^5 + c_2\omega_0^1) \wedge \omega_0^3 = 0, \\ &(\omega_0^0 - \omega_2^2 - \omega_3^3 + \omega_5^5 + c_2\omega_0^1) \wedge \omega_0^2 - (\omega_4^5 - c_3\omega_0^1 - 2\omega_3^2) \wedge \omega_0^3 = 0, \\ &\omega_4^\lambda \wedge \omega_0^2 + \omega_5^\lambda \wedge \omega_0^3 = 0, \\ &\omega_5^\lambda \wedge \omega_0^2 - \omega_4^\lambda \wedge \omega_0^3 = 0, \end{aligned} \tag{3.77}$$

where $\lambda = 6, \ldots, N$. System (3.76)–(3.77) contains $s_1 = 2N - 4$ independent equations that include the following independent characteristic forms:

$$\begin{aligned} &\Delta c_2, \ \Delta c_3, \ \omega_2^3 + \omega_3^2, \ \omega_2^2 - \omega_3^3, \\ &\omega_0^0 + \omega_4^4 + c_2\omega_0^1 - 2\omega_2^2, \omega_2^3 - \omega_3^2 + \omega_5^4 + c_3\omega_0^1, \\ &\omega_4^5 - c_3\omega_0^1 - 2\omega_2^3, \omega_0^0 - \omega_2^2 - \omega_3^3 + \omega_5^5 + c_2\omega_0^1, \\ &\omega_4^\lambda, \ \omega_5^\lambda, \quad \lambda = 6, \ldots, N. \end{aligned}$$

Their number is $q = 2N - 2$. The second character of the system is therefore $s_2 = q - s_1 = 2$, and the Cartan number $Q = s_1 + 2s_2 = 2N$. The number S of parameters on which the most general integral element depends is computed from the formula $S = 2q - s_1 = 2N$. Because $Q = S$, by the Cartan test, the system of Pfaffian equations (3.71), (3.73), and (3.75) is in involution, and its general integral manifold depends on two arbitrary functions of two variables.

In case (b), we have $b_2^\alpha = b_2 b^\alpha$ and $b_3^\alpha = b_3 b^\alpha$. Equations (3.74) can therefore be written in the form

$$\begin{aligned} \omega_2^\alpha &= b^\alpha (b_2 \omega_0^2 + b_3 \omega_0^3), \\ \omega_3^\alpha &= b^\alpha (b_3 \omega_0^2 - b_2 \omega_0^3). \end{aligned}$$

Consequently,

$$\begin{aligned} dA_2 &\equiv (b_2 \omega_0^2 + b_3 \omega_0^3) B \pmod{T_x}, \\ dA_3 &\equiv (b_3 \omega_0^2 - b_2 \omega_0^3) B \pmod{T_x}, \end{aligned}$$

where $B = b^\alpha A_\alpha$. We specialize our moving frame assuming $A_4 = B$. Then equations (3.74) take the form

$$\omega_2^4 = b_2 \omega_0^2 + b_3 \omega_0^3, \quad \omega_3^4 = b_3 \omega_0^2 - b_2 \omega_0^3, \tag{3.78}$$

$$\omega_2^\lambda = 0, \quad \omega_3^\lambda = 0, \tag{3.79}$$

where $\lambda = 5, \ldots, N$. Exterior differentiation of the last two equations gives the following quadratic equations:

$$\omega_2^4 \wedge \omega_4^\lambda = 0, \quad \omega_3^4 \wedge \omega_4^\lambda = 0.$$

Because by (3.78), 1-forms ω_2^4 and ω_3^4 are linearly independent, it follows from the last equations that

$$\omega_4^\lambda = 0, \quad \lambda = 5, \ldots, N.$$

This means that the variety X belongs to the four-dimensional space $\mathbb{P}^4$ spanned by the points A_0, A_1, A_2, A_3 and A_4. In case (b), the variety X is thus a hypersurface in the space $\mathbb{P}^4$, being defined in this space by the system of equations (3.71) (with $\alpha = 4$), (3.73), and (3.78).

We now investigate the consistency of the last system. For this purpose we apply exterior differentiation to equations (3.78). As a result, we obtain the following quadratic equations:

$$\begin{aligned} &(\Delta b_2 - 2(b_2 \omega_2^2 + b_3 \omega_2^3)) \wedge \omega_0^2 + \Delta b_3 \wedge \omega_0^3 = 0, \\ &\Delta b_3 \wedge \omega_0^2 - (\Delta b_2 - 2(b_2 \omega_3^3 - b_3 \omega_3^2)) \wedge \omega_0^3 = 0, \end{aligned} \tag{3.80}$$

where

$$\Delta b_2 = db_2 + b_2(\omega_0^0 + \omega_4^4) + (c_2 b_2 - c_3 b_3)\omega_0^1,$$
$$\Delta b_3 = db_3 + b_3(\omega_0^0 - \omega_2^2 - \omega_3^3 + \omega_4^4) + b_2(\omega_2^3 - \omega_3^2) + (c_2 b_3 + c_3 b_2)\omega_0^1.$$

The system of exterior equations (3.76) and (3.80) consists of $s_1 = 4$ independent equations. They contain $q = 6$ characteristic forms. As a result, the character $s_2 = q - s_1 = 2$, and the Cartan number $Q = s_1 + 2s_2 = 8$. The number S of parameters, on which the most general integral element depends, is also equal to 8. Because $Q = S$, by the Cartan test, the system of Pfaffian equations (3.71), (3.73), and (3.78) is in involution, and its general integral manifold depends on two arbitrary functions of two variables.

Thus, three-dimensional varieties X of rank two in $\mathbb{RP}^N$ that have no real singularities exist, in both cases (a) and (b), and a general integral manifold, defining such varieties, depends on two arbitrary functions of two variables.

Next, we show how one can construct parabolic varieties without singularities in a simply connected space $\mathbb{V}_c^N$ of constant curvature c.

Example 3.13. Suppose that a simply connected space $\mathbb{V}_c^N$ of constant curvature c is realized in a projective space $\mathbb{RP}^N$, and let G be its proper domain. If X is a three-dimensional parabolic variety of rank 2 in $\mathbb{RP}^N$ that has no real singularities, then the intersection $X \cap G$ is a variety having the same properties in V_c^N. Such varieties consequently also exist in V_c^N, and a general integral manifold of the system, defining such varieties, depends on two arbitrary functions of two variables.

Note that in Section 3.3, we constructed another example of a variety with a degenerate Gauss map without singularities in a real projective space $\mathbb{RP}^N$ (see Example 3.7).

NOTES

3.1. In the theory of partial differential equations, the Monge–Ampère equation is the equation of the form

$$rt - s^2 = ar + 2bs + ct + \phi, \tag{3.81}$$

where

$$r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2};$$

the coefficients a, b, c, and ϕ are functions of $x, y, z, p = \frac{\partial z}{\partial x}$; and $q = \frac{\partial z}{\partial y}$, and $z = z(x, y)$ is an unknown function (see, for example, the book Goursat [Go 42], pp. 47–62).

The case

$$rt - s^2 = 0 \tag{3.82}$$

is the most interesting for differential geometry. Equation (3.82) determines developable surfaces in a three-dimensional Euclidean space $\mathbb{E}^3$.

A generalization of equation (3.82) for a multidimensional space has the form

$$\det(u_{ij}) = 0, \quad i, j = 1, \dots, n, \tag{3.83}$$

where $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$. In the Euclidean space $\mathbb{E}^{n+1}$ (as well as in the affine apace $\mathbb{A}^{n+1}$ and the projective space $\mathbb{P}^{n+1}$), a function $u(x_1, \dots, x_n)$ satisfying equation (3.83) determines a hypersurface X with the equation $u = u(x_1, \dots, x_n)$ having a degenerate Gauss map. The rank of X equals the rank of the matrix (u_{ij}).

If the matrix (u_{ij}) has a constant rank $r < n$, then the hypersurface X carries plane generators L of dimension $n - r$ along which the tangent hyperplanes to X are constant. These plane generators L compose the Monge–Ampère foliation.

In this book we consider the Monge–Ampère foliations not only on hypersurfaces but also on varieties of codimension higher than one.

See more on connections between the Monge–Ampère equations and the geometry of manifolds in the papers [De 89] by Delanoë and [Mo 79] by Morimoto.

Ishikawa and Morimoto found the connection between varieties with degenerate Gauss maps and solutions of Monge–Ampère equations. In [IM 01], the authors proved that the rank r of a compact C^∞-hypersurface $X \subset \mathbb{RP}^N$ with a degenerate Gauss map is an even integer r satisfying the inequality $\frac{r(r+3)}{2} > N$, $r \neq 0$. In particular, if $r < 2$, then X is necessarily a projective hyperplane of $\mathbb{RP}^N$, and if $N = 3$ or $N = 5$, then a compact C^∞-hypersurface with a degenerate Gauss map is a projective hyperplane.

In our exposition of basic equations we follow the recently published paper by Akivis and Goldberg [AG 01a].

Theorem 3.1 is fundamental in the theory of varieties with degenerate Gauss maps. In some investigations, the authors take this property as the definition of such varieties—see, for example, the Akivis paper [A 57]. However, such varieties are usually defined in terms of reduced rank of the Gauss map. Then this property is proved—see, for example, Theorem 2.10 in the Griffiths and Harris paper [GH 79]; Theorem 4.1 in the book [AG 93] by Akivis and Goldberg (see also Theorem 1 in their recently published paper [AG 01a]); Landsberg's book [L 99] (§5); and the Linearity Theorem in Section 2.3 of the recently published book [FP 01] by Fischer and Piontkowski.

The proof of this theorem in the paper [HN 59] by Hartman and Nirenberg is based on the lemma on the constancy of a certain unique $(n-1)$-plane. This lemma was proved in the paper [CL 57] by Chern and Lashof. Sternberg [Ste 64] called this lemma the lemma of Chern–Lashof–Hartman–Nirenberg. A projective analogue of this lemma is our Theorem 3.1 (see also Theorem 4.1 in the book [AG 93] by Akivis and Goldberg and Theorem 1 in their paper [AG 01a]).

Theorem 3.2 proved at the end of Section 3.1 generalizes for varieties with degenerate Gauss maps Theorem 3.5 of the paper [GH 79] proved by Griffiths and Harris for tangentially nondegenerate varieties.

In Section 3.1 we assume that every plane generator L of a variety X with a degenerate Gauss map has at least one regular point. Otherwise, the Monge–Ampère foliation is degenerate, and we will not consider this case in the book.

3.2. Most of the results of this section are due to Akivis (see [A 57, 62]).

Note that a rather rough classification of two-dimensional and three-dimensional varieties with degenerate Gauss maps is given in the paper [GH 79] by Griffiths and Harris. On a classification of three-dimensional varieties with degenerate Gauss maps see the papers [Rog 97] by Rogora and [MT 02a] by Mezzetti and Tommasi.

Mezzetti and Tommasi [MT 02b] constructed an example of an irreducible two-dimensional algebraic variety (F) in the space $\mathbb{P}^4$ that generates a hypersurface X with a degenerate Gauss map of rank two. From the differential geometry point of view, the surface (F) in their example is separated into two parts (two separate surfaces) (F_1) and (F_2) by a curve Γ (see Figure 2.5). The hypersurface X is formed by the straight lines joining the points of (F_1) and (F_2) connected by the Laplace transform. On the curve Γ the Laplace transform is undetermined. In our opinion, the hypersurface X has two focal surfaces. But both focal surfaces are described by the same algebraic equation. Hence, while from the algebraic geometry point of view, the example constructed in [MT 02b] belongs to class b) indicated in the text, from the differential geometry point of view, this example belongs to class a).

3.3. Example 3.6 is new. The hypercubic in Example 3.7 is from the recent paper [WZ 02] by Wu and F. Zheng.

3.4. Sacksteder's hypersurface was considered by Sacksteder in [S 60]. It was the first example of a hypersurface with degenerate Gauss map of rank two without singularities in a Euclidean space $\mathbb{E}^4$. In 1995, Wu [Wu 95] published an example of a noncylindrical tangentially degenerate algebraic hypersurface in a Euclidean space $\mathbb{E}^4$ that has a degenerate Gauss map but does not have singularities. This example was constructed (but not published) by Bourgain (see also the papers [I 98, 99a, 99b] of Ishikawa). Theorem 3.9 was proved in the paper [AG 02a] by Akivis and Goldberg.

Note that Mori in his paper [M 94] claimed that he constructed an example of a noncylindrical hypersurface with degenerate Gauss map without singularities in a Euclidean space $\mathbb{E}^4$ (see also Ishikawa's papers [I 98, 99b] in which this result was mentioned). However, the authors of this book proved that the hypersurface in Mori's example is cylindrical. This was communicated to Mori, who recognized that his claim was wrong.

3.5. As we remarked in the Notes to Section 2.3, the notion of the index of relative nullity was introduced by Chern and Kuiper in [CK 52] (see also Kobayashi and Nomizu [KN 63], vol. 2, p. 348). This number is also called the Gauss defect of the system of second fundamental forms Φ^α of the variety X (see, for example, the book [FP 01], p. 89, by Fischer and Piontkowski).

Complete parabolic varieties in a Euclidean space $\mathbb{E}^n$ were studied by Borisenko in [Bor 82, 85]. In [Bor 92], he used the notion of parabolicity to formulate and prove a theorem on the unique determination of $V^m \subset \mathbb{E}^n$ from its Grassmann image.

Akivis [A 87a] recognized that the problem of finding singularities on complete parabolic varieties in a Riemannian space V_c^n of constant curvature and distinguishing those varieties that have no singularities is related not so much to the metric as to the projective structure of the spaces V_c^n. In this section we follow Akivis's paper [A 87a].

Chapter 4

Main Structure Theorems

In this chapter, in the projective space $\mathbb{P}^N$, we consider the basic types of varieties with degenerate Gauss maps: torsal varieties, hypersurfaces, and cones. For each of these types of varieties, we consider the structure of their focal images and find sufficient conditions for a variety with a degenerate Gauss map to belong to one of these types (for torsal varieties our condition is also necessary). In Theorems 4.3, 4.4, 4.5, 4.15, and 4.16, we establish connections between the types and the structure of focal images of varieties with degenerate Gauss maps of rank r. In Section 4.3, we consider varieties with degenerate Gauss maps in the affine space $\mathbb{A}^N$ and find a new affine analogue of the Hartman–Nirenberg cylinder theorem. In Section 4.4, we define and study new types of varieties with degenerate Gauss maps: varieties with multiple foci and their particular case, the so-called twisted cones. We also prove here existence theorems for some varieties with degenerate Gauss maps, for example, for twisted cones in $\mathbb{P}^4$ and $\mathbb{A}^4$ (Theorems 4.12 and 4.14) and establish a structure of twisted cones in $\mathbb{P}^4$ (Theorems 4.13). This structure allows us to find a procedure for construction of twisted cylinders in $\mathbb{A}^4$. In Section 4.5, we prove that varieties with degenerate Gauss maps that do not belong to one of the basic types considered in Sections 4.1–4.2 are foliated into varieties of basic types (Theorem 4.16). A classification of varieties X with degenerate Gauss maps presented in this chapter is based on the structure of the focal images F_L and Φ_L of X. In Section 4.6, we prove an embedding theorem for varieties with degenerate Gauss maps and find sufficient conditions for such a variety to be a cone (Theorems 4.18 and 4.19 in Section 4.6).

4.1 Torsal Varieties

As we saw earlier, in the projective space $\mathbb{P}^N$, there exist several types of varieties X of dimension $n < N$ with degenerate Gauss maps of rank $r < n$: torsal varieties (see Example 2.5), hypersurfaces with degenerate Gauss maps (see Example 2.7), cones (see Example 2.4), and twisted cones (see Section 3.4).

In this chapter, we establish a connection between the structure of focal images of a variety with a degenerate Gauss map and the structure of the variety itself.

First, we prove two lemmas.

Lemma 4.1. *Suppose that $l \geq 1$, and the focus hypersurface $F_L \subset L$ does not have multiple components. Then all matrices B^α can be simultaneously diagonalized, $B^\alpha = \operatorname{diag}(b^\alpha_{pp})$,*[1] *and the focus hypercone Φ_L decomposes into r bundles of hyperplanes Φ_p in $\mathbb{P}^N$ whose centers are the $(n+1)$-planes $T_L \wedge B_p$, where $B_p = b^\alpha_{pp} A_\alpha$ are points located outside of the tangent subspace T_L. The dimension $n+m$ of the osculating subspace T^2_L of the variety X along a generator L does not exceed $n+r$.*

Proof. Because the hypersurface $F_L \subset L$ does not have multiple components, a general straight line λ lying in L intersects F_L at r distinct points. We place the vertices A_0 and A_1 of our moving frame onto the line λ. By (3.21), the coordinates of the common points of λ and F_L are defined by the equation

$$\det(\delta^q_p x^0 + c^q_{1p} x^1) = 0.$$

Because the straight line λ intersects the hypersurface F_L at r distinct points, the preceding equation has r distinct roots. This implies that the matrix C_1 can be diagonalized, $C_1 = \operatorname{diag}\ (c^p_{1p})$ (no summation over p), and $c^p_{1p} \neq c^q_{1q}$ for $p \neq q$.

Next we write equations (3.9) for $a = 1$:

$$b^\alpha_{qp} c^p_{1p} = b^\alpha_{pq} c^q_{1q}.$$

Because $c^p_{1p} \neq c^q_{1q}$ and $b^\alpha_{pq} = b^\alpha_{qp}$, it follows that $b^\alpha_{pq} = 0$ for $p \neq q$. As a result, all matrices B^α can be simultaneously diagonalized, $B^\alpha = \operatorname{diag}\ (b^\alpha_{pp})$. Equation (3.24) takes the form

$$\prod_p (\xi_\alpha b^\alpha_{pp}) = 0,$$

and the focus hypercone Φ_L decomposes into r bundles of hyperplanes Φ_p in $\mathbb{P}^N$ whose axes are the $(n+1)$-planes $T_L \wedge B_p$, where $B_p = b^\alpha_{pp} A_\alpha$ are points located outside of the tangent subspace T_L. The osculating subspace T^2_L of the variety X along a generator L is the span of the tangent subspace T_L and the points $B_{l+1}, \ldots, B_n$. Thus, the dimension of this subspace does not exceed $n + r$. □

[1] Here and in what follows, we use a shorter notation for diagonal matrices: diag (b^α_{pp}) instead of diag $(b^\alpha_{l+1,l+1}, \ldots, b^\alpha_{nn})$ and diag (c^p_{ap}) instead of diag $(c^{l+1}_{a,l+1}, \ldots, c^n_{an})$.

Lemma 4.2. *Suppose that $m \geq 2$ and the focus hypercones Φ_L of a variety X do not have multiple components. Then all matrices C_a can be simultaneously diagonalized, $C_a = \mathrm{diag}\,(c^p_{ap})$, and the focus hypersurfaces F_L decompose into r plane components defined in L by the equations $x^0 + c^p_{ap}x^a = 0$ (no summation over p). The dimension $k = l - m^*$ of the characteristic subspace K_L is greater than or equal to $l - r$, where m^* is the number of linearly independent matrices C_a.*

Proof. Let ξ_{n+1} and ξ_{n+2} be two hyperplanes in general position that are tangent to a variety X along its generator L. They determine a pencil of tangent hyperplanes $\xi = \xi_{n+1} + \lambda\xi_{n+2}$. By (3.24), the intersection of this pencil with the focus hypercone Φ_L is defined by the equation

$$\det\,(b^{n+1}_{pq} + \lambda b^{n+2}_{pq}) = 0; \tag{4.1}$$

this equation is the λ-equation of the matrices B^{n+1} and B^{n+2} (see Böcher [Bö 07], Chapter XIII, no. 57). Because the focus hypercone Φ_L does not have multiple components and the pencil ξ is in general position with respect to this hypercone, equation (4.1) has r distinct roots. This implies that the matrices (b^{n+1}_{pq}) and (b^{n+2}_{pq}) can be simultaneously diagonalized (see, for example, Böcher [Bö 07], Chapter XIII, no. 58, Theorem 2),

$$B^{n+1} = \mathrm{diag}(b^{n+1}_{pp}), \;\; B^{n+2} = \mathrm{diag}(b^{n+2}_{pp})$$

and

$$\frac{b^{n+1}_{pp}}{b^{n+1}_{qq}} \neq \frac{b^{n+2}_{pp}}{b^{n+2}_{qq}} \;\text{ for }\; p \neq q. \tag{4.2}$$

Consider further equations (3.9) for $\alpha = n+1, n+2$. These equations and inequalities (4.2) imply that

$$c^p_{aq} = 0 \;\text{ for }\; p \neq q,$$

i.e., all matrices C_a are simultaneously diagonalized,

$$C_a = \;\mathrm{diag}\;(c^p_{ap}) \;\;\text{(no summation over } p).$$

As a result, equation (3.21) of the focus hypersurface F_L becomes

$$\prod_{p=l+1}^{n} (x^0 + c^p_{ap}x^a) = 0. \tag{4.3}$$

Thus, the hypersurface F_L decomposes into r plane components F_p defined in L by the equation

$$x^0 + c^p_{ap}x^a = 0 \;\;\text{(no summation over } p). \tag{4.4}$$

The characteristic subspace K_L (see Section 3.2.3) is the intersection of these plane components, and its dimension is $k = l - m^*$, where m^* is the number of linearly independent equations (4.4). Because $m^* \leq r$, we have $k \geq l - r$. □

We recall that a variety X with a degenerate Gauss map of rank r is torsal if it foliates into r families of simple (i.e., not multiple) torses.

Theorem 4.3. *A variety X with a degenerate Gauss map of rank $r > 1$ is torsal if and only if all its focus hypersurfaces F_L decompose into r simple $(l-1)$-planes F_p, $p = l+1, \ldots, n$, belonging to its plane generators L, and all its focus hypercones Φ_L decompose into r simple bundles Φ_p of hyperplanes with $(n+1)$-dimensional vertices that contain the tangent subspaces T_L of the variety X.*

Proof. Necessity: Suppose that a variety X is torsal of rank $r > 1$. Let τ_1 be one of the families of torses into which X foliates. This family is defined on X by the system of equations

$$\omega^{l+2} = \ldots = \omega^n = 0, \tag{4.5}$$

and the form ω^{l+1} is a basis form on torses of this family. A plane generator L of X is defined by the points $A_0, A_1, \ldots, A_l$ of a moving frame associated with X. By (3.5), (3.11), and (4.5), on τ_1, the differentials of these points have the form

$$\left\{\begin{array}{l} dA_0 = \omega_0^0 A_0 + \omega_a A_a + \omega^{l+1} A_{l+1}, \\ dA_a = \omega_a^0 A_0 + \omega_a^b A_b + c^p_{a,l+1}\omega^{l+1} A_p. \end{array}\right.$$

However, because on a torse, $\dim(L+dL) = l+1$ (see Example 2.4), the tangent subspaces to a torse must be determined by the points $A_0, A_1, \ldots, A_l, A_{l+1}$. As a result, we have $c^p_{a,l+1} = 0$ for $p = l+2, \ldots, n$.

Because by the theorem hypotheses, the variety X is torsal, i.e, it foliates into r families τ_p of torses, in a similar manner, we prove that in the matrices $C_a = (c^p_{aq})$, all nondiagonal entries vanish. Thus, all these matrices can be simultaneously diagonalized:

$$C_a = \operatorname{diag}\,(c^p_{ap}). \tag{4.6}$$

By means of (4.6), equation (3.21) determining a focus hypersurface F_L in a plane generator L takes the form

$$\prod_p (x^0 + c^p_{ap} x^a) = 0.$$

This proves that F_L decomposes into r hyperplanes defined in L by the equations

$$x^0 + c^p_{ap} x^a = 0. \tag{4.7}$$

These hyperplanes are tangent to the torses τ_p into which the variety X foliates. Because all these torses are simple, equations (4.7) for different values of p are not proportional. As a result, the rows of the matrix

$$C = \begin{pmatrix} 1 & c^{l+1}_{1,l+1} & \dots & c^{l+1}_{l,l+1} \\ 1 & c^{l+2}_{1,l+2} & \dots & c^{l+2}_{l,l+2} \\ \dots & \dots & \dots & \dots \\ 1 & c^{n}_{1,n} & \dots & c^{n}_{l,n} \end{pmatrix} \tag{4.8}$$

composed from eigenvalues of the matrices C_a are not pairwise proportional.

By (4.6), conditions (3.9) take the form

$$b^\alpha_{pq} c^p_{ap} = b^\alpha_{qp} c^q_{aq}.$$

Because the matrices $B^\alpha = (b^\alpha_{pq})$ are symmetric, it follows that

$$b^\alpha_{pq}(c^p_{ap} - c^q_{aq}) = 0. \tag{4.9}$$

Because the rows of the matrix C in (4.8) are not proportional, for any pair of indices $p, q, p \neq q$, there exists a value of a such that

$$c^p_{ap} \neq c^q_{aq}.$$

As a result, equation (4.9) implies that

$$b^\alpha_{pq} = 0 \;\; \text{for} \;\; p \neq q.$$

Thus the matrices B^α are also simultaneously diagonalized:

$$B^\alpha = \text{ diag } (b^\alpha_{pp}). \tag{4.10}$$

It follows from (4.10) that equation (3.24) of the focus hypercone Φ_L takes the form

$$\prod_p (\xi_\alpha b^\alpha_{pp}) = 0,$$

and this hypercone decomposes into r bundles of hyperplanes defined by the equations

$$\xi_\alpha b^\alpha_{pp} = 0.$$

The centers of these bundles are the $(n+1)$-planes $T_L \wedge B_p$, where $B_p = b^\alpha_{pp} A_\alpha$ are points not belonging to the subspace T_L. It is not difficult to prove that these $(n+1)$-planes are tangent to the torses $\widetilde{\tau}_p$ described by the subspace T_L when it moves along the torses $\tau_p \subset X$. Because the torses τ_p are simple, the torses $\widetilde{\tau}_p$ are also simple. Thus all the points B_p are distinct, and as a result, the columns of matrix (4.10) composed from eigenvalues of the matrix B_α are not proportional.

Sufficiency. Suppose that all focus hypersurfaces F_L of a variety X decompose into r simple hyperplanes $F_p \subset L$, and all its hypercones Φ_L decompose into r simple bundles Φ_p with the centers $T_L \wedge B_p$, where $B_p \notin T_L$. Then by Lemmas 4.1 and 4.2, all matrices C_a and B^α can be simultaneously diagonalized,

$$C_a = \operatorname{diag}\,(c^p_{ap}), \quad B^\alpha = \operatorname{diag}\,(b^\alpha_{pp}).$$

This implies that equations (3.11) and (3.4) take the form

$$\omega^p_a = c^p_{ap}\omega^p, \quad \omega^\alpha_p = b^\alpha_{pp}\omega^p, \tag{4.11}$$

where there is no summation over the index p.

Consider l-planes L defined by the points $A_0, A_1, \ldots, A_l$ of the moving frame associated with a variety X. By (2.67) and (3.7), we have

$$\begin{cases} dA_0 = \omega^0_0 A_0 + \omega^a_0 A_a + \omega^p A_p, \\ dA_a = \omega^0_a A_0 + \omega^b_a A_b + \sum\limits_p c^p_{ap}\omega^p A_p. \end{cases} \tag{4.12}$$

It follows that for

$$\omega^q = 0, \quad q \neq p, \tag{4.13}$$

there is no summation over p on the right-hand side of equations (4.12). Thus, the tangent subspace to the one-parameter family of rectilinear generators L defined on X by equations (4.13) is the plane $L \wedge A_p$ of dimension $l+1$. Therefore, this family of planes is a torse τ_p.

Thus, we have proved that the variety X foliates into r families of torses defined on X by the systems of equations (4.13). These families of torses are mutually simple because the $(l+1)$-planes $L \wedge A_p$ tangent to torses of these families passing through their common rectilinear generator L are simple. □

It is not difficult to prove that the families of the tangent subspaces T_L defined on X by the system of equations (4.13) also form torses $\widetilde{\tau}_p$ of dimension $n+1$.

An example of a torsal variety was considered in Section 2.4 (see Example 2.7); for $n = 3$, $r = 2$, this is a hypersurface $X = V^3_2 \subset \mathbb{P}^4$ (see Figure 2.5).

4.2 Hypersurfaces with Degenerate Gauss Maps

4.2.1 Sufficient Condition for a Variety with a Degenerate Gauss Map to be a Hypersurface in a Subspace of $\mathbb{P}^N$. First we consider the structure of the focal images of hypersurfaces with degenerate Gauss maps.

Theorem 4.4. *If a variety X with a degenerate Gauss map of dimension n and rank r is a hypersurface in a subspace $\mathbb{P}^{n+1} \subset \mathbb{P}^N$, then all its focus hypercones Φ_L are r-fold bundles of hyperplanes of the space $\mathbb{P}^N$ with the center $\mathbb{P}^{n+1}$.*

Proof. Suppose that $X \subset \mathbb{P}^{n+1} \subset \mathbb{P}^N$, and the subspace $\mathbb{P}^{n+1}$ is defined by the tangent subspace T_L of X and a point $B = b^\alpha A_\alpha$, $\alpha = n+1, \dots, N$, i.e., $\mathbb{P}^{n+1} = T_L \wedge B$. Then the variety X has only one independent second fundamental form

$$\Phi = b_{pq}\omega^p\omega^q, \quad p, q = l+1, \dots, n,$$

where $\det(b_{pq}) \neq 0$.

But, with respect to an arbitrary tangent hyperplane $\xi_\alpha x^\alpha = 0$ of the variety X, its second fundamental form can be written in the form

$$\Phi = \xi_\alpha b^\alpha_{pq}\omega^p\omega^q$$

(see (2.21)), where b^α_{pq} is the second fundamental tensor of the variety X. Thus,

$$\Phi(\xi) = \lambda(\xi)\Phi.$$

It follows that

$$\xi_\alpha b^\alpha_{pq} = \lambda(\xi) b_{pq}.$$

This implies that $\lambda(\xi) = \xi_\alpha b^\alpha$ and

$$b^\alpha_{pq} = b^\alpha b_{pq}.$$

As a result, equation (3.24) of the focus hypercones Φ_L of X takes the form

$$(\xi_\alpha b^\alpha)^r \cdot \det(b_{pq}) = 0.$$

Hence the focus hypercones Φ_L is an r-fold bundle of hyperplanes

$$b^\alpha\xi_\alpha = 0,$$

passing through the subspace $\mathbb{P}^{n+1} = T_L \wedge B$, where $B = b^\alpha A_\alpha$. □

The next theorem gives a sufficient condition for a variety $X \subset \mathbb{P}^N$ with a degenerate Gauss map to be a hypersurface in a subspace $\mathbb{P}^{n+1} \subset \mathbb{P}^N$.

Theorem 4.5. *Suppose that a variety X with a degenerate Gauss map of rank $r \geq 2$ and dimension n satisfies the conditions:*

(i) *On X, the conditions of Lemma 4.1 are satisfied, i.e., $l \geq 1$ and all focus hypersurfaces $F_L \subset L$ do not have multiple components.*

(ii) *All focus hypercones Φ_L are r-fold bundles of hyperplanes with $(n+1)$-dimensional centers P_L^{n+1} containing the tangent subspace T_L.*

Then the variety X is a hypersurface in a subspace $\mathbb{P}^{n+1} \subset \mathbb{P}^N$.

Proof. By Lemma 4.1 and condition (i) of the theorem, all matrices $B_\alpha = (b^\alpha_{pq})$ of the variety X can be simultaneously diagonalized,

$$B^\alpha = \operatorname{diag}\,(b^\alpha_{pp}),$$

and the focus hypercones Φ_L defined by equation (3.24) decompose into bundles of hyperplanes whose axes are the subspaces $T_L \wedge B_p$, where $B_p = b^\alpha_{pp} A_\alpha$.

But from condition (ii) it follows that the axes of these bundles coincide, and this implies that the eigenvalues b^α_{pp} of the tensors b^α_{pq} are proportional. As a result, the tensors b^α_{pq} themselves are proportional. The last condition can be expressed by the formula

$$b^\alpha_{pq} = b^\alpha b_{pq}, \tag{4.14}$$

where $\det\,(b_{pq}) \neq 0$, because the rank of the system of tensors b^α_{pq} is equal to r and $p, q = l+1, \ldots, n$, i.e., the indices p and q take r values.

Because condition (4.14) means that the variety$X$ has only one independent second fundamental form

$$\Phi = b_{pq} \omega^p \omega^q$$

and $r \geq 2$, then the Segre theorem (see Theorem 2.1 in Section 2.2.5) implies that the variety X is a hypersurface in a subspace $\mathbb{P}^{n+1} \subset \mathbb{P}^N$. □

The last result can be proved directly. In fact, it follows from (4.14) that

$$\omega^\alpha_p = b^\alpha b_{pq} \omega^q. \tag{4.15}$$

Taking exterior derivatives of equations (4.15), we obtain the following exterior quadratic equations:

$$(b_{pq} \nabla b^\alpha + b^\alpha \nabla b_{pq}) \wedge \omega^q = 0, \tag{4.16}$$

where

$$\begin{aligned} \nabla b^\alpha &= db^\alpha + b^\beta \omega^\alpha_\beta, \\ \nabla b_{pq} &= db_{pq} - b_{sq} \omega^s_p - b_{ps} \theta^s_q, \end{aligned}$$

and

$$\theta_q^p = \omega_q^p - \delta_q^p \omega_0^0 - c_{aq}^p \omega^a.$$

The 1-forms ω^q, $q = l+1, \ldots, n$, are basis forms on the variety X. However, they might not compose a complete basis of all 1-forms defined on this variety. We supplement the forms ω^q by 1-forms ω^u in such a way that the forms ω^q and ω^u compose a basis of the system of 1-forms occurring in equation (4.16). Then the decompositions of the forms ∇b^α and ∇b_{pq} occurring in (4.16) can be written as

$$\left\{\begin{array}{l} \nabla b^\alpha = b_p^\alpha \omega^p + t_u^\alpha \omega^u, \\ \nabla b_{pq} = b_{pqs} \omega^s + t_{pqu} \omega^u. \end{array}\right. \tag{4.17}$$

If we substitute (4.17) into (4.16) and equal to 0 the coefficients in the product of the independent forms ω^s and ω^u, we find that

$$b_{pq} b_s^\alpha - b_{ps} b_q^\alpha + b^\alpha (b_{pqs} - b_{psq}) = 0 \tag{4.18}$$

and

$$b_{pq} t_u^\alpha + b^\alpha t_{pqu} = 0. \tag{4.19}$$

It follows from (4.19) that

$$t_u^\alpha = t_u b^\alpha, \;\; t_{pqu} = -t_u b_{pq}, \tag{4.20}$$

where t_u are parameters. Note that equations (4.20) imply equations (4.19).

Contracting equations (4.18) with the tensor b^{pq}, which is the inverse tensor of b_{pq}, we find that

$$(r-1) b_s^\alpha + b^\alpha (b_{pqs} - b_{psq}) b^{pq} = 0.$$

Because by theorem hypotheses $r \geq 2$, it follows that

$$b_s^\alpha = b^\alpha b_s, \tag{4.21}$$

where

$$b_s = \frac{1}{r-1} (b_{pqs} - b_{psq}) b^{pq}.$$

As a result, the first equation of system (4.17) takes the form

$$\nabla b^\alpha = b^\alpha b_s \omega^s. \tag{4.22}$$

Next, we consider the subspace $\mathbb{P}^{n+1} = T_L \wedge B$, where $B = b^\alpha A_\alpha$. Differentiating the points $A_p \in T_L$ and applying formulas (3.5) and (4.14), we find that

$$dA_p \equiv b_{pq} \omega^q B \pmod{T_L}. \tag{4.23}$$

Differentiating point B and applying formulas (4.22), we obtain

$$dB \equiv b_q \omega^q B \pmod{T_L}. \tag{4.24}$$

Equations (4.23) and (4.24) mean that the subspace $\mathbb{P}^{n+1} = T_L \wedge B$ is fixed when we move along a generator $L \subset X$. Thus, $P_L^{n+1} = \mathbb{P}^{n+1} \subset \mathbb{P}^N$, and therefore, X is a hypersurface in $\mathbb{P}^{n+1} \subset \mathbb{P}^N$.

4.2.2 Focal Images of a Hypersurface with a Degenerate Gauss Map. Let us study the focal images of a hypersurface with a degenerate Gauss map of rank r in the space $\mathbb{P}^{n+1}$. On such a hypersurface, formulas (3.4) and (3.11) become

$$\omega_p^{n+1} = b_{pq}\omega^q, \ \ \omega_a^p = c_{aq}^p \omega^q, \tag{4.25}$$

where $b_{pq} = b_{qp}$ and $\det(b_{pq}) \neq 0$. Conditions (3.9) now take the form

$$b_{ps} c_{aq}^s = b_{qs} c_{ap}^s.$$

By (4.25), equation (3.24) of the focus hypercone Φ_L takes the form

$$\det(b_{pq}\xi_{n+1}) = 0.$$

It follows that $\xi_{n+1} = 0$, and the last equation defines the tangent subspace T_L of X. Thus, the focus hypercone Φ_L is reduced to its vertex T_L.

As in the general case (cf. (3.21)), the equation of the focus hypersurface $F_L \subset L$ has the form

$$\det(\delta_q^p x^0 + c_{aq}^p x^q) = 0,$$

and F_L is an algebraic hypersurface of degree r in a generator L. For $l \geq 2$, *in the general case* this hypersurface is indecomposable. For example, as we saw earlier, for the cubic symmetroid considered in Section 2.5.2, this hypersurface is a nondegenerate conic belonging to the two-dimensional generator L.

If all focus hypersurfaces F_L of a hypersurface X decompose into hypersurfaces F_L' and F_L'' of orders r' and r'' ($r' \geq 2, r'' \geq 2$, and $r' + r'' = r$), then the hypersurface X decomposes into two families of varieties X' and X'' of dimension $n' = l + r'$ and $n'' = l + r''$, respectively, and by Theorem 4.5, each of these varieties is a hypersurface in a subspace of dimension $n' + 1$ and $n'' + 1$, respectively.

If all focus hypersurfaces F_L of a hypersurface $X \subset \mathbb{P}^{n+1}$ decompose into r simple planes of dimension $l - 1$, then by Theorem 4.3, the hypersurface X is torsal, i.e., X foliates into r families of simple torses.

In Chapter 5, we will study in more detail the case when n-dimensional varieties with degenerate Gauss maps foliate into similar varieties of smaller dimensions.

4.2.3 Examples of Hypersurfaces with Degenerate Gauss Maps. As we noted in Section 2.5.1, a variety dual to a tangentially nondegenerate variety of dimension r in a projective space $\mathbb{P}^N$ is a tangentially degenerate hypersurface of rank r and dimension $n = N - 1$.

Another example of a hypersurface with a degenerate Gauss map is the cubic symmetroid considered in Section 2.5.2, where $N = 5, n = 4$, $r = 2$, $l = 2$.

Now we give a new example of a hypersurface with a degenerate Gauss map. This example generalizes the cubic symmetroid.

Example 4.6. Consider a hyperquadric Q in the space $\mathbb{P}^n$ defined by the equation

$$a_{uv}x^u x^v = 0, \quad a_{uv} = a_{vu}, \qquad u, v = 0, 1, \ldots, n. \tag{4.26}$$

For each hyperquadric Q, there is a corresponding point in the space $\mathbb{P}^N$, where $N = \frac{1}{2}(n+1)(n+2) - 1$. The coordinates of this point are the coefficients a_{uv} of equation (4.26). The degenerate hyperquadrics—the hypercones—are defined by the condition

$$\det(a_{uv}) = 0, \tag{4.27}$$

which determines a hypersurface $V^{N-1} \subset \mathbb{P}^N$. Because the degree of degeneracy of a hyperquadric can vary, in the space $\mathbb{P}^N$ we can consider the sequence of varieties defined by the equations

$$\operatorname{rank}(a_{uv}) = \rho, \tag{4.28}$$

where $2 \leq \rho < n$. Each term of this sequence defines a variety of hypercones with $(n-\rho+1)$-dimensional plane generators and an $(n-\rho)$-dimensional vertex in $\mathbb{P}^N$.

If $\rho = 1$, the variety (4.28) is a Veronese variety (see Section 1.5.2). In this case

$$a_{uv} = a_u a_v, \quad u, v = 0, 1, \ldots, n \tag{4.29}$$

(cf. (1.171)), and the hyperquadric (4.26) defined by the tensor a_{uv} becomes a double hyperplane $a_u x^u = 0$.

Let us study the structure of the determinant variety (4.28) in the general case $\rho = n - 1$. Such a variety is a hypersurface V^{N-1} in $\mathbb{P}^N$. To the points of V^{N-1}, there correspond the hypercones Q_0 with 0-dimensional vertex (a point) in $\mathbb{P}^N$.

We consider a family of moving frames $\{A^{uv}\}$ in $\mathbb{P}^N$, such that $A^{uv} = \alpha^u \alpha^v$, where α^u is a basis hyperplane of the space $(\mathbb{P}^n)^*$. Because the equations of infinitesimal displacement of a tangential moving frame in the space $(\mathbb{P}^n)^*$ have the form (1.79),

$$d\alpha^u = -\omega^u_v \alpha^v, \tag{4.30}$$

for the moving frames in the space $\mathbb{P}^N$ we get

$$dA^{uv} = -\omega_w^u A^{wv} - \omega_w^v A^{uw}. \tag{4.31}$$

Consider a family of hypercones Q with the common vertex A_0 in the space $\mathbb{P}^n$. The equation of this family has the form

$$a_{ij} x^i x^j = 0, \;\; i, j = 1, \ldots, n. \tag{4.32}$$

In the space $\mathbb{P}^N$, to this family of hypercones, there corresponds the subspace defined by the equations $a_{00} = 0, a_{0i} = 0$. This subspace is a plane generator L of the hypersurface V^{N-1}. The dimension l of this generator is equal to $l = \frac{n(n+1)}{2} - 1 = \frac{n^2+n-2}{2}$, and the set of all these generators depends on n parameters. So, the rank r of the hypercones Q is $r = \operatorname{rank} Q = n$.

In the space $\mathbb{P}^N$, the points A^{ij} of our moving frame lie on a generator L of the hypersurface V^{N-1}. Applying formulas (4.31), we calculate the differentials of the points A^{ij}:

$$dA^{ij} = -\omega_k^i A^{kj} - \omega_k^j A^{ki} - \omega_0^i A^{0j} - \omega_0^j A^{0i}. \tag{4.33}$$

Let $x = x_{ij} A^{ij}$ be a point of the generator L. Then

$$dx = (dx_{ij} - x_{ik}\omega_j^k - x_{kj}\omega_i^k) A^{ij} - 2x_{ij}\omega_0^i A^{0j}. \tag{4.34}$$

This equation shows that at all points of the generator L for which

$$\det(x_{ij}) \neq 0, \tag{4.35}$$

the tangent subspace T_x to the hypersurface V^{N-1} is determined by the points A^{ij} and A^{0j}. Hence this subspace is of dimension $N-1$ and is constant for all points $x \in L$ for which inequality (4.35) holds. Therefore, V^{N-1} is a hypersurface with a degenerate Gauss map of rank n.

At the points of the generator L_0 for which

$$\det(x_{ij}) = 0, \tag{4.36}$$

the dimension of the tangent subspace $T_x(V^{N-1})$ is reduced. Thus these points are foci of the generator, and singular points of the hypersurface V^{N-1}.

4.3 Cones and Affine Analogue of the Hartman–Nirenberg Cylinder Theorem

4.3.1 Structure of Focus Hypersurfaces of Cones. As we saw earlier, in the space $\mathbb{P}^N$, cones with $(l-1)$-dimensional vertices and with l-dimensional plane generators have degenerate Gauss maps (see Example 2.4 in Section 2.4). We now prove the following theorem describing the structure of focus hypersurfaces of such cones.

Theorem 4.7. *If a variety X with a degenerate Gauss map of dimension n and rank $r \geq 1$ is a cone with vertex of dimension $l-1$, then all its focus hypersurfaces F_L are r-fold $(l-1)$-dimensional planes belonging to its generator L.*

Proof. Suppose that X is a cone with an $(l-1)$-dimensional vertex S, where $l = n-r$, and l-dimensional plane generators L. We associate with X a family of moving frames such that the points $A_1, \ldots, A_l \in S$ and $A_0 \in L$. Because the vertex of the cone X is fixed, then on X, equations (3.5) take the form

$$\begin{aligned} dA_0 &= \omega_0^0 A_0 + \omega_0^a A_a + \omega^p A_p, \\ dA_a &= \qquad\qquad \omega_a^b A_b, \end{aligned}$$

where $a, b = 1, \ldots, l$; $p = l+1, \ldots, n$. It follows that $\omega_a^p = 0$, and all matrices C_a are zero matrices, $c_{aq}^p = 0$. As a result, equation (3.21) of the focus hypersurface $F_L \subset L$ becomes

$$\det (x^0 \delta_q^p) = 0,$$

i.e., $(x^0)^r = 0$, and the focus hypersurface F_L is an r-fold hyperplane $x^0 = 0$, which coincides with the vertex S of the cone X. □

The next theorem gives a sufficient condition for a variety $X \subset \mathbb{P}^N$ with a degenerate Gauss map to be a cone.

Theorem 4.8. *Suppose that X is a variety with a degenerate Gauss map of dimension n and rank $r \geq 2$ in the projective space $\mathbb{P}^N$, and X satisfies the following conditions:*

(i) *All focus hypersurfaces F_L are r-fold hyperplanes belonging to its plane generators L.*

(ii) *On X, the conditions of Lemma 4.2 are satisfied, i.e., $m \geq 2$, and all focus hypercones Φ_L do not have multiple components.*

Then the variety X is a cone with an $(l-1)$-dimensional vertex and l-dimensional plane generators.

Proof. By Lemma 4.2, all matrices $C_a = (c^p_{aq})$ of the variety X can be simultaneously diagonalized, $C_a = \operatorname{diag}(c^p_{ap})$, and its focus hypersurfaces F_L decompose into r hyperplanes defined by the equations

$$x^0 + c^p_{ap} x^a = 0.$$

But by condition (i) of the theorem, all these hyperplanes belonging to a generator L coincide. This implies that

$$c^p_{ap} = c^q_{aq} := c_a.$$

Thus, the entries of all matrices $C_a = (c^p_{aq})$ take the form

$$c^p_{aq} = c_a \delta^p_q.$$

The equations of the r-fold focus hyperplanes of the variety X can be written in the form

$$x^0 + c_a x^a = 0.$$

If we locate the points A_a of our moving frame of X in this hyperplane, then we obtain $c_a = 0$, and as a result, we have

$$c^p_{aq} = 0, \quad a = 1, \dots, l,$$

for all $p, q = l+1, \dots, n$. Therefore, by (3.11), we obtain

$$\omega^p_a = 0. \tag{4.37}$$

Taking exterior derivatives of equation (4.37), we arrive at the exterior quadratic equation

$$\omega^0_a \wedge \omega^p_0 = 0.$$

But because $r \geq 2$, and the forms ω^p_0 are linearly independent, it follows from the above quadratic equations that

$$\omega^0_a = 0. \tag{4.38}$$

Now from equations (3.5) and (4.38) it follows that

$$dA_a = \omega^b_a A_b,$$

and the $(l-1)$-plane $S = A_1 \wedge A_2 \wedge \ldots \wedge A_l$ is fixed. Thus, the variety X is an n-dimensional cone with the vertex S and l-dimensional plane generators $L = A_0 \wedge A_1 \wedge \ldots \wedge A_l$. □

4.3.2 Affine Analogue of the Hartman–Nirenberg Cylinder Theorem. The Hartman–Nirenberg cylinder theorem in an $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ was first proved by Hartman and Nirenberg in their joint paper [HN 59]. This theorem states the following.

Theorem 4.9 (The Hartman–Nirenberg Cylinder Theorem). *Let $X \subset \mathbb{E}^{n+1}$ be a connected, complete, C^2, orientable hypersurface in an $(n+1)$-dimensional space $\mathbb{E}^{n+1}$. If X is of constant zero curvature, then it is an $(n-1)$-cylinder (i.e., an n-dimensional cylinder with $(n-1)$-dimensional generators erected over a curve) in the sense that X has a parameterization (in the large) of the form*

$$\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}) = \sum_{i=1}^{n-1} \boldsymbol{a}_i x^i + \boldsymbol{b}(x^n) \text{ for all } \boldsymbol{x} = (x^1, \ldots, x^n), \tag{4.39}$$

where $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}$ are constant vectors in $\mathbb{E}^{n+1}$; $\boldsymbol{b}(x^n)$ is a vector-valued function of a variable x^n of class C^2 in $\mathbb{E}^{n+1}$; and $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{n-1}$, $\partial \boldsymbol{b}/\partial x^n$ is a set of orthonormal vectors.

In the proof of this theorem, Hartman and Nirenberg first proved that the vanishing of the Gaussian curvature implies that the rank $r(x)$ of the Gauss map of X does not exceed one, $r(x) \leq 1$. If $r(x) = 0$, then X is a hyperplane. In the case $r(x) = 1$, X is an $(n-1)$-cylinder that can be parameterized as indicated in equation (4.39).

The proof of this theorem in the paper [HN 59] by Hartman and Nirenberg is based on the lemma on the constancy of a certain unique $(n-1)$-plane. This lemma was proved in the paper [CL 57] by Chern and Lashof. Sternberg [Ste 64] called this lemma the lemma of Chern–Lashof–Hartman–Nirenberg. A projective analogue of this lemma is our Theorem 3.1 (see also Theorem 1 in the paper [A 87a] by Akivis, Theorem 4.1 in the book [AG 93] by Akivis and Goldberg, and Theorem 1 in their paper [AG 01a]).

Note that in [HN 59] and [Ste 64], the authors obtain an $(n-1)$-cylinder, i.e., a cylinder in $\mathbb{E}^{n+1}$ with $(n-1)$-dimensional plane generators erected over a curve. The reason they did not get an $(n-r)$-cylinder, i.e., an n-dimensional cylinder in $\mathbb{E}^{n+1}$ with $(n-r)$-dimensional plane generators erected over an r-dimensional manifold, where $r = 1, \ldots, n-1$, is that the vanishing of the Gaussian curvature implies that the rank $r(x)$ of the Gauss map of X does not exceed one.

The Hartman–Nirenberg cylinder theorem is of affine nature. In fact, the notion of a cylinder appearing in the theorem conclusion is an affine notion. As to the theorem hypotheses, although the notion of the Gaussian curvature is not affine, the notion of the rank of the Gauss map, which is fundamental in the proof of this theorem and whose boundedness, $r(x) \leq 1$, is implied by the vanishing of the Gaussian curvature, is even of projective nature. This is why it is interesting to consider an *affine analogue* of the Hartman–Nirenberg cylinder theorem.

We recall that in an affine space $\mathbb{A}^N$, an *l-cylinder* X over the field of complex or real numbers is defined as a smooth n-dimensional submanifold bearing l-dimensional plane generators, $l < n$, which are parallel to each other. An l-cylinder is a variety with a degenerate Gauss map of rank $r = n - l$. In an affine space $\mathbb{A}^N, N > n$, an l-cylinder can be defined by a parametric equation

$$\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}) = \sum_{i=1}^{l} \boldsymbol{a}_i x^i + \boldsymbol{b}(x^{l+1}, \dots, x^n) \text{ for all } \boldsymbol{x} = (x^1, \dots, x^n), \tag{4.40}$$

where $\boldsymbol{a}_i$ are constant vectors in $\mathbb{A}^N$, $\boldsymbol{b}(x^{l+1}, \dots, x^n)$ is a vector-valued function of $r = n - l$ variables defining in $\mathbb{A}^N$ a *director manifold* Y of the cylinder X, and the vectors $\boldsymbol{a}_i$ and $\boldsymbol{b}_p = \frac{\partial \boldsymbol{b}}{\partial x^p}$ are linearly independent.

In this section we prove the following *affine analogue of the Hartman–Nirenberg cylinder theorem.*

Theorem 4.10 (An Affine Analogue of the Hartman–Nirenberg Cylinder Theorem). *Let X^n be a smooth, projectively complete, connected variety with a degenerate Gauss map of constant rank r, $2 \leq r \leq n-1$, without singularities in a real or complex affine space $\mathbb{A}^N$, $N - n \geq 2$. Suppose that in the pencil of the second fundamental forms of X, there are two forms defining a regular pencil, all eigenvalues of which are distinct. Then the variety X is a cylinder with l-dimensional plane generators, $l = n - r \geq 2$, and an r-dimensional tangentially nondegenerate director variety Y. In $\mathbb{A}^N$ such a cylinder can be defined by parametric equation* (4.40)*.*

Proof. We enlarge the space $\mathbb{A}^N$ to a projective space $\mathbb{P}^N$ by attaching a hyperplane at infinity $\mathbb{P}^{N-1}_\infty$. So we have

$$\mathbb{P}^N = \mathbb{A}^N \cup \mathbb{P}^{N-1}_\infty.$$

Consider the submanifold X described in the theorem in the space $\mathbb{P}^N$. This submanifold $X \subset \mathbb{P}^N$ satisfies all conditions of Theorem 4.8 Thus, X is a cone with an $(l-1)$-dimensional vertex in $\mathbb{P}^N$. But because X is projectively complete in the space $\mathbb{A}^N$, all its singular points are located at a hyperplane at infinity $\mathbb{P}^{N-1}_\infty$. Thus, X is a cylinder in $\mathbb{A}^N$. □

4.4 Varieties with Degenerate Gauss Maps with Multiple Foci and Twisted Cones

4.4.1 Basic Equations of a Hypersurface of Rank r with r-Multiple Focus Hyperplanes. In Section 4.3, in a projective space $\mathbb{P}^N$, we considered varieties X with degenerate Gauss maps of dimension n and rank r with the following two properties:

(i) Their focus hypersurfaces F_L degenerate into r-fold hyperplanes.

(ii) Their system of second fundamental forms possesses at least two forms whose λ-equation has r distinct roots.

We have proved that such varieties X are cones in the space $\mathbb{P}^N$ with a vertex of dimension $l-1$, where $l = n - r$.

In this section we also consider varieties X with degenerate Gauss maps of dimension n and rank $r \geq 2$ with r-fold focus hyperplanes but we assume that all their second fundamental forms are proportional, i.e., for each pair of second fundamental forms of X, their λ-equation has an r-multiple eigenvalue.

Because we assume $r \geq 2$, the generalized Segre theorem (see Theorem 2.1 in Section 2.2.5) implies that such varieties are hypersurfaces in a subspace $\mathbb{P}^{n+1}$. We shall prove that such hypersurfaces can differ from cones.

Consider a hypersurface X with a degenerate Gauss map of dimension n and rank r whose focus hypersurfaces F_L are r-fold hyperplanes of dimension $l-1$, where $l = n - r$ is the dimension of the Monge–Ampère foliation on X. We associate a family of moving frames with X in such a way that the point $A_0 = x$ is a regular point of a generator L, the points A_a, $a = 1, \ldots, l$, belong to the r-fold focus hyperplane F_L, the points A_p, $p = l+1, \ldots, n$, lie in the tangent hyperplane $T_L(X)$, and the point A_{n+1} is situated outside of this hyperplane. As a result of such frame specialization, basic equations (3.4) and (3.11) of the variety X take the form

$$\omega_p^{n+1} = b_{pq}\omega^q, \;\; \omega_a^p = c_{aq}^p \omega^q, \quad p, q = l+1, \ldots, n, \tag{4.41}$$

where $B = (b_{pq})$ is a nondegenerate symmetric $(r \times r)$-matrix. Because the points A_a, $a = 1, \ldots, l$, belong to the r-fold focus $(l-1)$-plane F_L, the equation of F_L is

$$(x^0)^r = 0.$$

However, in the general case the focus hypersurface F_L of the generator L is determined by the equation

$$\det\left(\delta_q^p x^0 + c_{aq}^p x^a\right) = 0$$

(see (3.21)). Hence, we have

$$\det(\delta_q^p x^0 + c_{aq}^p x^a) = (x^0)^r$$

It follows that each of the matrices C_a has an r-multiple eigenvalue 0, and as a result, each of these matrices is nilpotent. We assume that each of the matrices C_a has the form

$$C_a = (c_{aq}^p), \text{ where } c_{aq}^p = 0 \text{ for } p \geq q. \tag{4.42}$$

Thus, rank $C_a \leq r - 1$. It follows that all matrices C_a are nilpotent. Denote by r_1 the maximal rank of matrices from the bundle $C = x^a C_a$, $r_1 \leq r - 1$.

It is obvious that this form is sufficient for all F_L to be r-fold hyperplanes. Wu and F. Zheng [WZ 02] (see also Piontkowski [Pio 01, 02b]) proved that this form is also necessary for the ranks $r = 2, 3, 4$ and different values of the maximum rank r_1 of matrices of the bundle $x^a C_a$. For $r \leq 4$, condition (4.42) is also necessary for $F_L \subset L$ to be an r-fold hyperplane. However, Wu and F. Zheng in [WZ 02] gave also a counterexample which proves that for $r \geq 5$, the form (4.42) is not necessary for all F_L to be r-fold hyperplanes.

A single second fundamental form of X at its regular point $x = A_0$ can be written as

$$\Phi_0 = b_{pq}\omega^p\omega^q.$$

This form is of rank r. At singular points A_a belonging to an r-multiple focus hyperplane F_L, the second fundamental form of the hypersurface X has the form

$$\Phi_a = b_{ps}c_{aq}^s\omega^p\omega^q, \tag{4.43}$$

where $(b_{ps}c_{aq}^s)$ is a symmetric matrix. The maximal rank of matrices from the bundle $\Phi = x^a\Phi_a$ is also equal to $r_1 \leq r - 1$.

4.4.2 Hypersurfaces with Degenerate Gauss Maps of Rank r with a One-Dimensional Monge–Ampère Foliation and r-Multiple Foci. Let A_0A_1 be a leaf of the Monge–Ampére foliation, let A_0 be a regular point of this leaf, and let A_1 be its r-multiple focus. Then in equations (4.41), we have $a, b = 1$; $p, q = 2, \ldots, n$, and these equations become

$$\omega_p^{n+1} = b_{pq}\omega^q, \;\; \omega_1^p = c_q^p\omega^q. \tag{4.44}$$

By our assumption (4.42), the matrix $C = (c_q^p)$ has the form

$$C = \begin{pmatrix} 0 & c_3^2 & \ldots & c_n^2 \\ \multicolumn{4}{c}{\dotfill} \\ 0 & 0 & \ldots & c_n^{n-1} \\ 0 & 0 & \ldots & 0 \end{pmatrix}. \tag{4.45}$$

We will assume that rank $C = r - 1 = n - 2$. Then in the matrix C, the coefficients $c^p_{p+1} \neq 0$. As to the matrix $B = (b_{pq})$, by the relation

$$BC = CB \tag{4.46}$$

(cf. (3.12), p. 94), this matrix has the form

$$B = \begin{pmatrix} 0 & \dots & 0 & b_{2,n} \\ 0 & \dots & b_{3,n-1} & b_{3,n} \\ \dots & \dots & \dots & \dots \\ b_{n,2} & \dots & b_{n,n-1} & b_{nn} \end{pmatrix}, \tag{4.47}$$

and rank $B = n - 1$. In addition, by (4.46), the entries of the matrices B and C are connected by certain bilinear relations implied by (4.46).

By (4.44), (4.45), and (4.47), on the hypersurface X, we have the equation

$$\omega^n_1 = 0. \tag{4.48}$$

Because on the hypersurface X equations (2.5) and (3.3) also hold, the differentials of the points A_0 and A_1 take the form

$$\begin{aligned} dA_0 &= \omega^0_0 A_0 + \omega^1_0 A_1 + \omega^2_0 A_2 + \dots + \omega^{n-1}_0 A_{n-1} + \omega^n_0 A_n, \\ dA_1 &= \omega^0_1 A_0 + \omega^1_1 A_1 + \omega^2_1 A_2 + \dots + \omega^{n-1}_1 A_{n-1}. \end{aligned} \tag{4.49}$$

In equations (4.49), the forms $\omega^2_1, \omega^3_1, \dots, \omega^{n-1}_1$ are linearly independent, and by (4.44) and (4.45), they are expressed in terms of the basis forms $\omega^3, \dots, \omega^n$ only. The following cases can occur:

1) The 1-form ω^0_1 is independent of the forms $\omega^3, \dots, \omega^n$, and hence also of the forms $\omega^2_1, \dots, \omega^{n-1}_1$. In this case, the r-multiple focus A_1 of the rectilinear generator L describes a focus variety G of dimension $r = n-1$. The variety G is of codimension two in the space $\mathbb{P}^{n+1}$ in which the hypersurface X is embedded. The tangent subspace $T_{A_1}(G)$ is defined by the points $A_1, A_0, A_2, \dots, A_{n-1}$. At the point A_1, the variety G has two independent second fundamental forms. We can find these two forms by finding the second differential of the point A_1:

$$d^2 A_1 \equiv \omega^p_1 \omega^n_p A_n + \omega^p_1 \omega^{n+1}_p A_{n+1} \pmod{T_{A_1}(G)}.$$

Thus, we have

$$\Phi^n_1 = \omega^p_1 \omega^n_p, \quad \Phi^{n+1}_1 = \omega^p_1 \omega^{n+1}_p.$$

The second of these forms coincides with the second fundamental form Φ_1 of the hypersurface X at the point A_1. By (4.45), if $\omega^3 = \dots = \omega^n = 0$, the

1-forms $\omega_1^p = 0$. Hence the quadratic forms Φ_1^n and Φ_1^{n+1} vanish on the focal variety G. Therefore, the direction $A_1 \wedge A_0$ is an asymptotic direction on the variety G.

2) The 1-form ω_1^0 is a linear combination of the forms $\omega_1^2, \ldots, \omega_1^{n-1}$, and hence also of the forms $\omega^3, \ldots, \omega^n$. In this case, the focus A_1 of the rectilinear generator L describes a focus variety G of dimension $n-2$, and its tangent subspace $T_{A_1}(G)$ is a hyperplane in the space $A_0 \wedge A_1 \wedge A_2 \wedge \ldots \wedge A_{n-1}$. For $\omega_1^2 = \ldots = \omega_1^{n-1} = 0$, the point A_1 is fixed, and the straight line $L = A_1 \wedge A_0$ describes a two-dimensional cone with vertex A_1. This cone is called the *fiber cone*. The hypersurface X foliates into an $(n-2)$-parameter family of such fiber cones. It is called a *twisted cone with rectilinear generators*.

In Section 4.4.3, for $n = 3$ we will prove that a fiber cone is a pencil of straight lines. Most likely this is true for any n.

3) Suppose that an $(n-2)$-dimensional focus variety G of the hypersurface X belongs to a hyperplane $\mathbb{P}^n$ of the space $\mathbb{P}^{n+1}$. We can take this hyperplane as the hyperplane at infinity $\mathbb{P}^n_\infty$ of the space $\mathbb{P}^{n+1}$. As a result, the space $\mathbb{P}^{n+1}$ becomes an affine space $\mathbb{A}^{n+1}$. In this case, the hypersurface X becomes a *twisted cylinder* in $\mathbb{A}^{n+1}$, which foliates into an $(n-2)$-parameter family of two-dimensional cylinders with rectilinear generators. The hypersurface X with a degenerate Gauss map is not a cylinder in $\mathbb{A}^{n+1}$ and does not have singularities in this space. Thus, this hypersurface is an affinely complete hypersurface in $\mathbb{A}^{n+1}$, which is not a cylinder. An example of such a hypersurface in the space $\mathbb{A}^4$ was considered by Sacksteder and Bourgain (see Sacksteder [S 60], Wu [Wu 95], Ishikawa [I 98, 99a, 99b], Akivis and Goldberg [AG 02a], and Section 3.4).

Note also that hypersurfaces with degenerate Gauss maps in the space $\mathbb{P}^{n+1}$ considered in this section are *lightlike hypersurfaces* which were studied in detail in the papers [AG 98b; 98c]) by Akivis and Goldberg. We will consider them in Section 5.1.

4.4.3 Hypersurfaces with Degenerate Gauss Maps with Double Foci on Their Rectilinear Generators in the Space $\mathbb{P}^4$. As an example, we consider hypersurfaces X with degenerate Gauss maps of rank $r = 2$ in the space $\mathbb{P}^4$ that have a single double focus F on each rectilinear generator $L = A_0 \wedge A_1$. With respect to a first-order frame, the basic equations of X are

$$\omega_0^4 = 0, \;\; \omega_1^4 = 0. \tag{4.50}$$

The basis forms of X are ω_0^2 and ω_0^3. By (4.44), (4.45), and (4.47), with respect to a second-order frame, we have the following equations

$$\left\{\begin{array}{ll} \omega_2^4 = \phantom{b_{32}\omega_0^2+} b_{23}\omega_0^3, & \omega_1^2 = c_3^2\omega_0^3, \\ \omega_3^4 = b_{32}\omega_0^2 + b_{33}\omega_0^3, & \omega_1^3 = 0, \end{array}\right. \tag{4.51}$$

where $b_{23} = b_{32} \neq 0$ and $c_3^2 \neq 0$. As a result, matrices B and C take the forms

$$B = \begin{pmatrix} 0 & b_{23} \\ b_{23} & b_{33} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & c_3^2 \\ 0 & 0 \end{pmatrix}.$$

The differentials of the points A_0 and A_1 are

$$\begin{aligned} dA_0 &= \omega_0^0 A_0 + \omega_0^1 A_1 + \omega_0^2 A_2 + \omega_0^3 A_3, \\ dA_1 &= \omega_1^0 A_0 + \omega_1^1 A_1 + \omega_1^2 A_2 \end{aligned}$$

(cf. (4.39)). The point $A_1 = F_L$ is a single focus of a rectilinear generator L.

Exterior differentiation of equations (4.51) gives the following exterior quadratic equations:

$$-2b_{23}\omega_2^3 \wedge \omega_0^2 + \Delta b_{23} \wedge \omega_0^3 = 0, \tag{4.52}$$

$$\Delta b_{23} \wedge \omega_0^2 + \Delta b_{33} \wedge \omega_0^3 = 0, \tag{4.53}$$

$$-(\omega_1^0 + c_3^2\omega_2^3) \wedge \omega_0^2 + \Delta c_3^2 \wedge \omega_0^3 = 0, \tag{4.54}$$

$$(\omega_1^0 - c_3^2\omega_2^3) \wedge \omega_0^3 = 0, \tag{4.55}$$

where

$$\begin{aligned} \Delta b_{23} &= db_{23} + b_{23}(\omega_0^0 - \omega_2^2 - \omega_3^3 + \omega_4^4) - b_{33}\omega_2^3, \\ \Delta b_{33} &= db_{33} + b_{33}(\omega_0^0 - 2\omega_3^3 + \omega_4^4) + b_{32}c_3^2\omega_0^1 - b_{32}\omega_3^2, \\ \Delta c_3^2 &= dc_3^2 + c_3^2(\omega_0^0 - \omega_1^1 + \omega_2^2 - \omega_3^3). \end{aligned}$$

From equations (4.52) and (4.55), it follows that the forms ω_2^3 and ω_1^0 are linear combinations of the basis forms ω_0^2 and ω_0^3. Three cases are possible:

1) $\omega_1^0 \wedge \omega_0^3 \neq 0$. Because by (4.51), this implies that $\omega_1^0 \wedge \omega_1^2 \neq 0$, it follows that the focus A_1 describes a two-dimensional focal surface G^2. The tangent plane to G^2 at the point A_1 is $T_{A_1}(G) = A_1 \wedge A_0 \wedge A_2$, and the straight line $L = A_0 \wedge A_1$ is tangent to G^2 at A_1.

2)

$$\omega_1^0 \wedge \omega_0^3 = 0. \tag{4.56}$$

In this case, the point A_1 describes a focal line G^1, and the straight line $L = A_0 \wedge A_1$ intersects this line G^1 at the point A_1. The hypersurface X

foliates into a one-parameter family of two-dimensional cones and is a twisted cone.

3) The osculating hyperplane of the curve G^1 is fixed.

We consider these three cases in detail.

1) We prove an existence theorem for this case applying the Cartan test (see Section 1.2.6).

Theorem 4.11. *Hypersurfaces X of rank two in the space $\mathbb{P}^4$, for which the single focus of a rectilinear generator L describes a two-dimensional surface, exist, and the general solution of the system defining such hypersurfaces depends on one function of two variables. The direction A_1A_0 is an asymptotic direction on the surfaces G^2, and the hypersurface X is formed by asymptotic tangents to the surfaces G^2.*

Proof. On a hypersurface in question, the inequality $\omega_1^0 \wedge \omega_0^3 \neq 0$ holds as well as the exterior quadratic equations (4.52)–(4.55). The latter equations contain five forms ω_2^3, Δb_{23}, Δb_{33}, ω_1^0, and Δc_3^2 that are different from the basis forms ω_0^2 and ω_0^3. So, we have $q = 5$.

The character s_1 of the system under investigation is equal to the number of independent exterior quadratic equations (4.52)–(4.55). Thus, we have $s_1 = 4$. As a result, the second character of the system is $s_2 = q - s_1 = 1$. Therefore, the Cartan number $Q = s_1 + 2s_2 = 6$.

We now calculate the number of parameters on which the most general integral element of the system under investigation depends. Applying Cartan's lemma to equations (4.52) and (4.53), we find that

$$\left\{\begin{aligned} -2b_{23}\omega_2^3 &= b_{222}\omega^2 + b_{223}\omega^3, \\ \Delta b_{23} &= b_{232}\omega^2 + b_{233}\omega^3, \\ \Delta b_{33} &= b_{332}\omega^2 + b_{333}\omega_0^3. \end{aligned}\right. \tag{4.57}$$

Because the coefficients of the basis forms on the right-hand sides of (4.57) are symmetric with respect to the lower indices, the number of independent coefficients on the right-hand sides of (4.57) is $S_1 = 4$.

Equation (4.55) implies that

$$\omega_1^0 = c_3^2\omega_2^3 + \lambda\,\omega_0^3. \tag{4.58}$$

We substitute this expression into equation (4.54). As a result, we obtain

$$-2(c_3^2\omega_2^3 + \lambda\,\omega_0^3) \wedge \omega_0^2 + \Delta c_3^2 \wedge \omega_0^3 = 0. \tag{4.59}$$

It follows from (4.59) that the 1-form Δc_3^2 is a linear combination of the basis forms. We write its expression in the form

$$\Delta c_3^2 = \mu\,\omega_0^2 + \nu\,\omega_0^3. \tag{4.60}$$

Because $b_{23} \neq 0$, we can find the form ω_2^3 from the first equation of system (4.57). Substituting this expression and (4.60) into equation (4.59), we find that

$$\left(\frac{c_3^2 b_{223}}{b_{23}} - \lambda\right)\omega_0^3 \wedge \omega_0^2 + \mu\,\omega_0^2 \wedge \omega_0^3 = 0.$$

This implies that

$$\mu = \frac{c_3^2 b_{223}}{b_{23}} - \lambda.$$

Thus, there are only two independent coefficients in decompositions (4.58) and (4.60), $S_2 = 2$.

As a result, we have $S = S_1 + S_2 = 6$, and $S = Q$. Applying the Cartan test, we conclude that the system under investigation is in involution, and its general solution depends on one function of two variables. □

Next, we find the second fundamental forms of the two-dimensional focal surface G^2 of the hypersurface X with a degenerate Gauss map. To this end, we compute

$$d^2 A_1 \equiv (\omega_1^0\omega_0^3 + \omega_1^2\omega_2^3)A_3 + \omega_1^2\omega_2^4 A_4 \pmod{T_{A_1}(G^2)}.$$

Thus, the second fundamental forms of G^2 are

$$\Phi_1^3 = \omega_1^0\omega_0^3 + \omega_1^2\omega_2^3, \;\; \Phi_1^4 = \omega_1^2\omega_2^4.$$

The direction A_1A_0 is defined on G_2 by the equation $\omega_1^2 = 0$. By (4.51), this equation is equivalent to the equation $\omega_0^3 = 0$. Thus, in this direction the second fundamental forms Φ_1^3 and Φ_1^4 vanish:

$$\Phi_1^3 \equiv 0 \pmod{\omega_0^3}, \;\; \Phi_1^4 \equiv 0 \pmod{\omega_0^3},$$

and the direction A_1A_0 is an asymptotic direction on the focal surface G^2.

2) We prove the following existence theorem for the twisted cones.

Theorem 4.12. *If condition* (4.56) *is satisfied, then the double focus A_1 of the generator $A_0 \wedge A_1$ of the variety X describes the focal curve, and X is a twisted cone. In the space $\mathbb{P}^4$, the twisted cones exist, and the general solution of the system defining such cones depends on five functions of one variable.*

Proof. In this case, the point A_1 describes the focal line G^1. By (4.56), we must enlarge the system of equations (4.51) by the equation

$$\omega_1^0 = a\,\omega_0^3. \tag{4.61}$$

Equation (4.61) is equivalent to equation (4.56). The 1-form ω_0^3 is a basis form on the focal line G^1. By (4.61), equation (4.55) takes the form

$$\omega_2^3 \wedge \omega_0^3 = 0.$$

This equation is equivalent to (4.56). It follows that

$$\omega_2^3 = b\,\omega_0^3. \tag{4.62}$$

Now equations (4.52) and (4.54) become

$$(\Delta b_{23} + 2b_{23}\, b\,\omega_0^2) \wedge \omega_0^3 = 0, \tag{4.63}$$

$$(\Delta c_3^2 + (a + b\,c_3^2)\,\omega^2) \wedge \omega_0^3 = 0. \tag{4.64}$$

Equation (4.53) remains the same.

Taking exterior derivatives of equations (4.61) and (4.62), we obtain the exterior quadratic equations

$$(da + a(2\omega_0^0 - \omega_1^1 - \omega_3^3) + c_3^2\,\omega_2^0 + ab\,\omega_0^2) \wedge \omega_0^3 = 0, \tag{4.65}$$

$$(db + b(\omega_0^0 - \omega_2^2) + b_{23}\,\omega_4^3 + b\,\omega_0^2) \wedge \omega_0^3 = 0. \tag{4.66}$$

Now the system of exterior quadratic equations consists of equations (4.53), (4.63)–(4.66). Thus, we have $s_1 = 5$. In addition to the basis forms ω_0^2 and ω_0^3, these exterior equations contain the forms Δb_{23}, Δb_{33}, Δc_3^2, Δa, and Δb, where

$$\Delta a = da + a(2\omega_0^0 - \omega_1^1 - \omega_3^3) + c_3^2\omega_2^0 \tag{4.67}$$

and

$$\Delta b = db + b(\omega_0^0 - \omega_2^2) + b_{23}\omega_4^3.$$

The number of these forms is $q = 5$. Thus, $s_2 = q - s_1 = 0$, and the Cartan number $Q = s_1 = 5$. If we find the forms Δb_{23}, Δb_{33}, Δc_3^2, Δa, and Δb from the system of equations (4.53), (4.63)–(4.66), we see that the most general integral element of the system under investigation (i.e., the dimension S of the space of integral elements over a point) depends on $S = 5$ parameters. Thus, $S = Q$, the system under investigation is in involution, and its general solution depends on five functions of one variable. □

Consider the focal curve G^1 of the twisted cone $X^3 \subset \mathbb{P}^4$ described by the point A_1. We have

$$dA_1 = \omega_1^1 A_1 + (c_3^2 A_2 + aA_0)\,\omega_0^3.$$

The point $\widetilde{A}_2 = c_3^2 A_2 + aA_0$ along with the point A_1 define a tangent line to G^1. Because $c_3^2 \neq 0$, we can specialize our moving frame by locating its vertex A_2 at $\widetilde{A}_2$ and by normalizing the frame by means of the condition $c_3^2 = 1$ (see Section 1.4). Then we obtain

$$dA_1 = \omega_1^1 A_1 + \omega_0^3 A_2.$$

In addition, the conditions

$$a = 0, \;\; c_3^2 = 1$$

are satisfied. These conditions and equations (4.51), (4.61), (4.64), and (4.67) imply that

$$\omega_1^2 = \omega_0^3, \;\; \omega_1^0 = 0, \tag{4.68}$$

$$\Delta c_3^2 = \omega_0^0 - \omega_1^1 + \omega_2^2 - \omega_3^3, \tag{4.69}$$

$$\Delta a = \omega_2^0. \tag{4.70}$$

After this specialization, the straight line $A_1 \wedge A_2$ becomes the tangent to the focal line G^1.

Now equations (4.64) and (4.65) take the form

$$\begin{aligned}&(\omega_0^0 - \omega_1^1 + \omega_2^2 - \omega_3^3 + b\,\omega_0^2) \wedge \omega_0^3 = 0,\\ &\omega_2^0 \wedge \omega_0^3 = 0.\end{aligned}$$

It follows from the last equation that

$$\omega_2^0 = c\,\omega_0^3. \tag{4.71}$$

Note also that equation (4.66) shows that because $b_{23} \neq 0$, the quantity b can be reduced to 0 by means of the form ω_4^3 (see Section 1.4). As a result, equation (4.62) takes the form

$$\omega_2^3 = 0, \tag{4.72}$$

and because $b_{23} \neq 0$, equation (4.66) becomes

$$\omega_4^3 \wedge \omega_0^3 = 0. \tag{4.73}$$

It follows from (4.73) that

$$\omega_4^3 = f\,\omega_0^3. \tag{4.74}$$

Differentiating the point A_2 and applying (4.51), (4.71), and (4.72), we obtain

$$dA_2 = \omega_2^2 A_2 + \omega_2^1 A_1 + (cA_0 + b_{23} A_4)\, \omega_0^3.$$

The 2-plane $\alpha = A_1 \wedge A_2 \wedge (cA_0 + b_{23} A_4)$ is the *osculating plane* of the line G^1 at the point A_1.

We place the point A_4 of our moving frame into the plane α and make a normalization $b_{23} = 1$. As a result, we have $c = 0$ and

$$\omega_2^0 = 0, \;\; \omega_2^4 = \omega_0^3. \tag{4.75}$$

Now, the plane α is defined as $\alpha = A_1 \wedge A_2 \wedge A_4$, and the differential of A_2 becomes

$$dA_2 = \omega_2^2 A_2 + \omega_2^1\, A_1 + \omega_0^3\, A_4.$$

Taking the exterior derivative of the first of two equations (4.75), we obtain

$$\omega_4^0 \wedge \omega_0^3 = 0,$$

and this implies that

$$\omega_4^0 = g\, \omega_0^3. \tag{4.76}$$

By means of equations (4.72) and (4.76), we find that

$$dA_4 = \omega_4^4 A_4 + \omega_4^1 A_1 + \omega_4^2 A_2 + (f A_3 + g A_0)\, \omega_0^3. \tag{4.77}$$

Equation (4.77) means that the 3-plane

$$\beta = A_1 \wedge A_2 \wedge A_4 \wedge (f A_3 + g A_0)$$

is the *osculating hyperplane* of the focal line G^1.

Taking exterior derivatives of equations (4.74) and (4.76), we find the following exterior quadratic equations:

$$(df + f(\omega_0^0 - \omega_4^4)) \wedge \omega_0^3 = 0, \tag{4.78}$$

and

$$(dg + g(2\omega_0^0 - \omega_3^3 - \omega_4^4) - f\omega_3^0) \wedge \omega_0^3 = 0. \tag{4.79}$$

Applying the analytic method of specialization of moving frames (see Section 1.4), we can prove that by means of the secondary forms $\omega_0^0 - \omega_4^4$ and ω_3^0, we can reduce the quantities f and g to the following values:

$$f = 1, \;\; g = 0,$$

As a result, equations (4.74) and (4.76) become

$$\omega_4^3 = \omega_0^3, \ \ \omega_4^0 = 0, \tag{4.80}$$

and the osculating hyperplane β of G^1 becomes

$$\beta = A_1 \wedge A_2 \wedge A_4 \wedge A_3.$$

Substituting the values $f = 1$ and $g = 0$ into equations (4.78) and (4.79), we obtain

$$(\omega_0^0 - \omega_4^4) \wedge \omega_0^3 = 0 \tag{4.81}$$

and

$$\omega_3^0 \wedge \omega_0^3 = 0. \tag{4.82}$$

Note that equations (4.81) and (4.82) could also be obtained by exterior differentiation of equations (4.80).

After this specialization, we obtain the following system of equations defining the twisted cones X in the space $\mathbb{P}^4$:

$$\begin{cases} \omega_2^4 = \omega_0^3, & \omega_3^4 = \omega^2, \\ \omega_1^2 = \omega_0^3, & \omega_1^3 = 0, \\ \omega_1^0 = 0, & \omega_2^3 = 0, \\ \omega_2^0 = 0, & \omega_2^4 = \omega^3, \\ \omega_4^3 = \omega_0^3, & \omega_4^0 = 0. \end{cases} \tag{4.83}$$

Note that in addition to all specializations made earlier, in equations (4.83), we also made a specialization $b_{33} = 0$ that can be achieved by means of the secondary form $\omega_0^1 - \omega_3^2$ (see the third equation in system (4.57), and the expression for Δb_{33} on p. 155, where $c_3^2 = 1$ and $b_{23} = 1$ as a result of the specializations made on pp. 158 and 159).

Taking exterior derivatives of equations (4.83), we find the following exterior quadratic equations:

$$\begin{cases} (\omega_0^0 - \omega_2^2 - \omega_3^3 + \omega_4^4) \wedge \omega_0^3 = 0, \\ (\omega_0^0 - \omega_2^2 - \omega_3^3 + \omega_4^4) \wedge \omega_0^2 + (\omega_0^1 - \omega_3^2) \wedge \omega_0^3 = 0, \\ (\omega_0^0 - \omega_1^1 + \omega_2^2 - \omega_3^3) \wedge \omega_0^3 = 0, \\ (\omega_0^0 - \omega_4^4) \wedge \omega_0^3 = 0, \\ \omega_3^0 \wedge \omega_0^3 = 0. \end{cases} \tag{4.84}$$

The exterior differentiation of the remaining five equations of system (4.83) leads to identities.

The system of equations (4.84) is equivalent to the system of equations (4.53), (4.63)–(4.66) from which it is obtained as a result of specializations. For the system of equations (4.84), as it was for the original system of equations (4.53), (4.63)–(4.66), we have

$$q = 5, \;\; s_1 = 5, \;\; s_2 = 0 \;\text{ and }\; Q = S = 5.$$

The system is in involution, and its solution exists and depends on five functions of one variable.

Formulas (4.83) and (4.84) allow us to prove the following theorem:

Theorem 4.13. *A twisted cone X in the space $\mathbb{P}^4$ foliates into a one-parameter family of pencils of straight lines whose centers are located on its focal line G^1 and whose planes are tangent to G^1.*

Proof. We consider the structure of the fiber cones of a twisted cone $X \subset \mathbb{P}^4$. The fiber cones C on X are defined by the system

$$\omega_0^3 = 0. \tag{4.85}$$

By (4.85) and (4.83), we have

$$dA_0 = \omega_0^0 A_0 + \omega_0^1 A_1 + \omega_0^2 A_2. \tag{4.86}$$

It follows that the plane $A_0 \wedge A_1 \wedge A_2$ is tangent to the fiber cone C along its generator $L = A_0 \wedge A_1$. By (4.85) and (4.83), the differential of the point A_2 is

$$dA_2 = \omega_2^1 A_1 + \omega_2^2 A_2, \tag{4.87}$$

and by (4.85), we also have

$$dA_1 = \omega_1^1 A_1. \tag{4.88}$$

Equations (4.86), (4.87), and (4.88) prove that the tangent plane $\gamma = A_0 \wedge A_1 \wedge A_2$ to the fiber cone C is fixed when the generator $L = A_0 \wedge A_1$ moves along C. It follows that a fiber cone C is simply a pencil of straight lines with center at A_1 located at the plane γ. □

Note that in the example of Sacksteder–Bourgain hypersurface (see Section 3.4), we have seen the same situation as in Theorem 4.13. However, in Theorem 4.13 we proved this fact for the general case.

Now we prove the converse statement: A general smooth one-parameter family of two-dimensional planes $\gamma(t)$ in the space $\mathbb{P}^4$ forms a three-dimensional twisted cone X. In fact, such a family envelopes a curve G^1, whose point A is the common point of the planes $\gamma(t)$ and $\gamma(t + dt)$, i.e.,

$$A(t) = \gamma(t) \cap \gamma(t + dt).$$

The point $A(t)$ and the plane $\gamma(t)$ define a pencil $(A, \gamma)(t)$ of straight lines with center $A(t)$ and plane $\gamma(t)$. The set of these pencils forms a three-dimensional ruled surface X with rectilinear generators L belonging to the pencils $(A, \gamma)(t)$. Moreover, the tangent space $T(X)$ is constant along a rectilinear generator L. Hence the rank of the variety X equals two.

Because the dimension of the Grassmannian $\mathbb{G}(2, 4)$ of two-dimensional planes in the space $\mathbb{P}^4$ is equal to six (see Section 1.4, p. 42), one-parameter family of such planes depends on five functions of one variable. This number coincides with the arbitrariness of existence of twisted cones in $\mathbb{P}^4$ that we computed earlier by investigating a system defining a twisted cone (see Theorem 4.12, p. 157).

3) Next we find under what condition a twisted cone becomes a twisted cylinder. This condition is equivalent to a condition under which the osculating hyperplane β of the focal curve G^1 is fixed, when the point A_1 moves along G^1. Because $\beta = A_1 \wedge A_2 \wedge A_3 \wedge A_4$ and

$$dA_3 = \omega_3^0 A_0 + \omega_3^1 A_1 + \omega_3^2 A_2 + \omega_3^3 A_3 + \omega_3^4 A_4,$$

the condition in question has the form

$$\omega_3^0 = 0. \tag{4.89}$$

If we take the fixed osculating hyperplane β of G^1 as the hyperplane at infinity H_∞ of the space $\mathbb{P}^4$, then $\mathbb{P}^4$ becomes an affine space $\mathbb{A}^4$. Then the hypersurface X becomes a twisted cylinder $\widetilde{X}$, which by Theorem 4.13, *foliates into a one-parameter family of planar pencils of parallel straight lines.* The hypersurface X does not have singularities in the space $\mathbb{A}^4$ and is a *complete smooth noncylindrical hypersurface of rank two.*

It is easy to prove the existence of twisted cylinders in the space $\mathbb{A}^4$.

Theorem 4.14. *Twisted cylinders in the space $\mathbb{A}^4$ exist, and the general solution of the system defining such cylinders depends on four functions of one variable.*

Proof. In fact, a twisted cylinder in $\mathbb{A}^4$ is defined by the system of equations (4.83) and (4.89). By (4.89), the last equation of system (4.84) becomes an identity. Exterior differentiation of (4.89) leads to an identity too. Thus, in the system of exterior quadratic equations (4.84), only four equations are independent. Thus, $s_1 = 4$, and equations (4.84) contain only four 1-forms that are different from the basis forms. Hence $q = 4$. Therefore,

$$s_2 = q - s_1 = 0, \quad Q = s_1 + 2s_2 = 4.$$

Equations (4.84) imply that $S = 4$. Because $Q = S$, the system is in involution, and its general solution depends on four functions of one variable. □

In conclusion, we indicate a construction defining the general twisted cylinders in the space $\mathbb{A}^4$. Let $\mathbb{P}^3$ be an arbitrary hyperplane in the projective space $\mathbb{P}^4$, and let G^1 be an arbitrary curve in $\mathbb{P}^3$. Consider a family of planes $\gamma(t)$ that are tangent to the curve G^1 but do not belong to $\mathbb{P}^3$, such that two infinitesimally close planes $\gamma(t)$ and $\gamma(t + dt)$ of this family do not belong to a three-dimensional subspace of the space $\mathbb{P}^4$. Then these two planes have only one common point $A(t) = \gamma(t) \cap \gamma(t + dt)$ belonging to G^1, and the planes $\gamma(t)$ form a twisted cone in the space $\mathbb{P}^4$. If we take the hyperplane $\mathbb{P}^3$ as the hyperplane at infinity of $\mathbb{P}^4$, then the space $\mathbb{P}^4$ becomes an affine space $\mathbb{A}^4$, and a twisted cone formed by the planes $\gamma(t)$ becomes a twisted cylinder in $\mathbb{A}^4$. Such a construction was considered by Akivis in his paper [A 87a].

4.4.4 The Case $n = 3$ (Continuation). In Section 3.2.5 we gave a complete classification of three-dimensional varieties X of rank two with degenerate Gauss maps in the case when each rectilinear generator L of X has two different foci F_1 and F_2. We indicated there five classes a), b), c), d), and e) of such varieties.

In this section we present a complete classification of three-dimensional varieties X of rank two with degenerate Gauss maps in the case when each rectilinear generator L of X has a double focus $F_1 = F_2 = F$.

If $F_1 = F_2 = F$, then the following three cases are possible:

f) If a double focus F describes a two-dimensional surface $V^2 = (F)$, then V^2 has a four-dimensional osculating subspace and bears one family of asymptotic lines, and a variety X is a union of tangents to a family of asymptotic lines of V^2 (cf. Theorem 4.11, p. 155).

g) If a double focus G describes a space curve $\gamma = (F)$, then a variety X is a twisted cone formed by plane pencils of straight lines whose centers belong to the curve γ and whose planes π are tangent to γ (cf. Theorem 4.13, p. 161).

   One can also say that in this case a variety X is a band, i.e., is the union of planes π that are tangent (not osculating) planes to the curve γ (the support curve of the band). For definition of a band see the books [Bl 21] (§33) or [Bl 50] (§21) by Blaschke or the book [AG 93] (Section 7.6) by Akivis and Goldberg.

h) If a double focus F is fixed, then X is a cone with vertex F.

4.5 Reducible Varieties with Degenerate Gauss Maps

4.5.1 Some Definitions. We saw in Section 3.1.2 that the system of matrices C_a and B^α are associated with a variety X with a degenerate Gauss map of rank r. This system is said to be *reducible* if these matrices can be simultaneously reduced to a block diagonal form:

$$C_a = \text{diag}\,(C_{a1}, \ldots, C_{as}), \;\; B^\alpha = \text{diag}\,(B^{\alpha 1}, \ldots, B^{\alpha s}), \tag{4.90}$$

where C_{at} and $B^{\alpha t}$, $t = 1, \ldots, s$, are square matrices of orders r_t, and $r_1 + r_2 + \ldots + r_s = r$. If such a decomposition of matrices is not possible, the system of matrices C_a and B^α is called *irreducible.* If $r_1 = r_2 = \ldots = r_s = 1$, then the system of matrices C_a and B^α is called *completely reducible.*

A variety X with a degenerate Gauss map of rank r is said to be *reducible, irreducible*, or *completely reducible* if for any values of parameters $u \in M$, the matrices C_a and B^α are reducible, irreducible, or completely reducible, respectively.

4.5.2 The Structure of the Focal Images of Reducible Varieties with Degenerate Gauss Maps. Equations (3.21) and (3.24) of focal images of a variety X with a degenerate Gauss map of rank r imply the following proposition describing the structure of the focus hypersurfaces F_L and the focus hypercones Φ_L of a reducible variety X.

Proposition 4.15. *Suppose that a variety X with a degenerate Gauss map of rank r is reducible. Then each of its focus hypersurfaces $F_L \subset L$ decomposes into s components F_t of dimension $l-1$ each and degrees $r_1, r_2, \ldots, r_s$, and each of its focus hypercones Φ_L decomposes into s hypercones Φ_t of the same degrees $r_1, r_2, \ldots, r_s$; $r_1 + r_2 + \ldots + r_s = r$, and with the same vertex T. In particular, if X is completely reducible, then a focus hypersurface F_L decomposes into r hyperplanes, and a focus hypercone Φ_L decomposes into r bundles of hyperplanes with $(n+1)$-dimensional axes.*

Proof. We assume that the index t takes only two values, $t = 1, 2$, $r = r_1 + r_2$, and the indices p and q have the following values:

$$p_1, q_1 = l+1, \ldots, l+r_1, \;\; p_2, q_2 = l+r_1+1, \ldots, n.$$

Then equations (3.11) and (3.4) become

$$\begin{cases} \omega_a^{p_1} = c_{aq_1}^{p_1} \omega^{q_1}, & \omega_{p_1}^\alpha = b_{p_1 q_1}^\alpha \omega^{q_1}, \\ \omega_a^{p_2} = c_{aq_2}^{p_2} \omega^{q_2}, & \omega_{p_2}^\alpha = b_{p_2 q_2}^\alpha \omega^{q_2}, \end{cases} \tag{4.91}$$

and the matrices C_a and B^α are reduced to the form

$$C_a = \begin{pmatrix} C_{a1} & 0 \\ 0 & C_{a2} \end{pmatrix}, \quad B^\alpha = \begin{pmatrix} B^{\alpha 1} & 0 \\ 0 & B^{\alpha 2} \end{pmatrix},$$

where

$$\begin{aligned} C_{a1} &= (c^{p_1}_{aq_1}) \quad B^{\alpha 1} = (b^\alpha_{p_1 q_1}), \\ C_{a2} &= (c^{p_2}_{aq_2}) \quad B^{\alpha 2} = (b^\alpha_{p_2 q_2}) \end{aligned}$$

are irreducible matrices. As a result, the equation of the focus hypersurface F_L of a generator L takes the form

$$\det \begin{pmatrix} \delta^{p_1}_{q_1} x_0 + c^{p_1}_{aq_1} x^{a_1} & 0 \\ 0 & \delta^{p_2}_{q_2} x^0 + c^{p_2}_{aq_2} x^{a_2} \end{pmatrix} = 0,$$

and the equation of the focus hypercone Φ_L with vertex T_L takes the form

$$\det \begin{pmatrix} \xi_\alpha b^\alpha_{p_1 q_1} & 0 \\ 0 & \xi_\alpha b^\alpha_{p_2 q_2} \end{pmatrix} = 0.$$

Thus the focus hypersurface F_L decomposes into two $(l-1)$-dimensional components F_1 and F_2 defined by the equations

$$\begin{aligned} F_1 &: \det(\delta^{p_1}_{q_1} x^0 + c^{p1}_{aq_1} x^{a_1}) = 0, \\ F_2 &: \det(\delta^{p_2}_{q_2} x^0 + c^{p2}_{aq_2} x^{a_2}) = 0 \end{aligned}$$

of degrees r_1 and r_2.

The focus hypercone Φ_L also decomposes into two components Φ_1 and Φ_2 defined by the equations

$$\begin{aligned} \Phi_1 &: \det(\xi_\alpha b^\alpha_{p_1 q_1}) = 0, \\ \Phi_2 &: \det(\xi_\alpha b^\alpha_{p_2 q_2}) = 0 \end{aligned}$$

of degrees r_1 and r_2.

The proof of Proposition 4.15 for any number of components is similar to the above proof. □

4.5.3 The Structure Theorems for Reducible Varieties with Degenerate Gauss Maps. In this subsection we prove the main theorems of this section.

Theorem 4.16. *Suppose that a variety X is reducible and its matrices $B^{\alpha t}$ and C_{it} defined in* (4.90) *are of order r_t, $t = 1, \dots, s$. Then X is foliated into s families of $(l + r_t)$-dimensional varieties of rank r_t with l-dimensional plane generators. For $r_t = 1$, these varieties are torses, and for $r_t \geq 2$, they are irreducible varieties described in Theorem* 4.4 *and* 4.5.

Proof. We again assume that the index t takes only two values, $t = 1, 2$, $r = r_1 + r_2$, and the indices p and q have the following values:

$$p_1, q_1 = l + 1, \dots, l + r_1, \quad p_2, q_2 = l + r_1 + 1, \dots, n.$$

Then equations (3.11) and (3.4) become (4.91).

Exterior differentiation of equations (4.91) gives

$$\nabla c^{p_1}_{aq_1} \wedge \omega^{q_1} + (c^{s_2}_{aq_2} \omega^{p_1}_{s_2} - c^{p_1}_{as_1} \theta^{s_1}_{q_2}) \wedge \omega^{q_2} = 0, \tag{4.92}$$

$$\nabla b^{\alpha}_{p_1 q_1} \wedge \omega^{q_1} - (b^{\alpha}_{s_2 q_2} \omega^{s_2}_{p_1} + b^{\alpha}_{p_1 s_1} \theta^{s_1}_{q_2}) \wedge \omega^{q_2} = 0, \tag{4.93}$$

$$\nabla c^{p_2}_{aq_2} \wedge \omega^{q_2} + (c^{s_1}_{aq_1} \omega^{p_2}_{s_1} - c^{p_2}_{s_2 i} \theta^{s_2}_{q_1}) \wedge \omega^{q_1} = 0, \tag{4.94}$$

$$\nabla b^{\alpha}_{p_2 q_2} \wedge \omega^{q_2} - (b^{\alpha}_{s_1 q_1} \omega^{s_1}_{p_2} + b^{\alpha}_{p_2 s_2} \theta^{s_2}_{q_1}) \wedge \omega^{q_1} = 0, \tag{4.95}$$

where

$$\begin{aligned}
\nabla c^{p_1}_{aq_1} &= dc^{p_1}_{aq_1} - c^{p_1}_{bq_1} \omega^b_a + c^{s_1}_{aq_1} \omega^{p_1}_{s_1} - c^{p_1}_{as_1} \theta^{s_1}_{q_1}, \\
\nabla b^{\alpha}_{p_1 q_1} &= db^{\alpha}_{p_1 q_1} + b^{\beta}_{p_1 q_1} \omega^{\alpha}_{\beta} - b^{\alpha}_{s_1 q_1} \omega^{s_1}_{p_1} - b^{\alpha}_{p_1 s_1} \theta^{s_1}_{q_1}, \\
\nabla c^{p_2}_{aq_2} &= dc^{p_2}_{aq_2} - c^{p_2}_{bq_2} \omega^b_a + c^{s_2}_{q_2 a} \omega^{p_2}_{s_2} - c^{p_2}_{as_2} \theta^{s_2}_{q_2}, \\
\nabla b^{\alpha}_{p_2 q_2} &= db^{\alpha}_{p_2 q_2} + b^{\beta}_{p_2 q_2} \omega^{\alpha}_{\beta} - b^{\alpha}_{s_2 q_2} \omega^{s_2}_{p_2} - b^{\alpha}_{p_2 s_2} \theta^{s_2}_{q_2};
\end{aligned}$$

as earlier, we use the notation

$$\theta^p_q = \omega^p_q - \delta^p_q \omega^0_0 - c^p_{aq} \omega^a.$$

Consider the system of equations

$$\omega^{q_1} = 0 \tag{4.96}$$

on the variety X. Its exterior differentiation gives

$$\omega^{q_2} \wedge \theta^{q_1}_{q_2} = 0, \tag{4.97}$$

where $\theta^{q_1}_{q_2} = \omega^{q_1}_{q_2} - \delta^{q_1}_{q_2} \omega^0_0 - c^{q_1}_{aq_2} \omega^a$. It follows from (4.97) that the conditions of complete integrability of equations (4.96) have the form

$$\theta^{q_1}_{q_2} = l^{q_1}_{q_2 s_2} \omega^{s_2}, \tag{4.98}$$

where $l^{q_1}_{q_2 s_2} = l^{q_1}_{s_2 q_2}$.

By (4.96), the system of equations (4.92) takes the form

$$(c^{s_2}_{iq_2}\omega^{p_1}_{s_2} - c^{p_1}_{is_1}\theta^{s_1}_{q_2}) \wedge \omega^{q_2} = 0, \tag{4.99}$$

where $i = \{0, a\}$ and $c^p_{0q} = \delta^p_q$. Suppose that the component F_1 of the focus hypersurface F_L does not have multiple components. Assuming that $l \geq 1$, we write equations (4.99) for two different values of the index i, for example, for $i = 0, 1$. Because the matrices $(c^{p_1}_{as_1})$ and $(c^{p_2}_{as_2})$ are not proportional, then it follows from (4.99) that two terms occurring in (4.99) vanish separately. In particular, this means that

$$c^{p_1}_{is_1}\theta^{s_1}_{q_2} \wedge \omega^{q_2} = 0. \tag{4.100}$$

Because the number of linearly independent forms among the 1-forms $\omega^{p_1}_i$ connected with the basis forms by relations (4.91) is equal to the number of linearly independent forms ω^{q_1} (i.e., it is equal to r_1), then it follows from (4.100) that

$$\theta^{s_1}_{q_2} \wedge \omega^{q_2} = 0.$$

But the last equations coincide with equations (4.97) and are conditions of complete integrability of (4.96). Thus, the variety X foliates into an r_1-parameter family of varieties of dimension $l + r_2$ and of rank r_2, and these varieties belong to the types described in Theorems 4.4 and 4.5.

In a similar way, one can prove the complete integrability of equations $\omega^{q_2} = 0$ on the variety X. Thus the variety X foliates also into an r_2-parameter family of varieties of dimension $l + r_1$ and of rank r_1.

By induction over s, we can prove the result, which we have proved for $s = 2$ components, for the case of any number s of components. □

Note that the torsal varieties described in Theorem 4.3 are completely reducible, and the varieties X described in Theorems 4.4 and 4.5 are irreducible varieties.

The following theorem follows from Theorem 4.8 and the theorems proved in Sections 4.1–4.2 and 4.4.

Theorem 4.17. *Suppose that X is a variety with a degenerate Gauss map in the space $\mathbb{P}^N$,* $\dim X = n$, $\operatorname{rank} X = r < n$, *for which all focus hypersurfaces F_L have components F^*_L of degree $r^* < r$. Then X foliates into $(r - r^*)$-parameter family of varieties X^* of dimension $n^* = r^* + l$, where $l = n - r$ is the dimension of plane generators of the variety X. Moreover,*

(a) *If each of the focus hypersurfaces F^*_L of a variety X^* decomposes into r^* simple hyperplanes, then all varieties X^* are torses.*

(b) *If* $r^* \geq 2$ *and the focus hypersurfaces* F_L^* *of a variety* X^* *do not decompose, then all varieties* X^* *are hypersurfaces in the space* $\mathbb{P}^{n^*+1}$*, where* $n^* = r^* + l$.

(c) *If* $r^* \geq 2$ *and the focus hypercones* Φ_L^* *of a variety* X^* *do not decompose, then all varieties* X^* *are cones with* $(l-1)$*-dimensional vertices.*

(d) *If* $l = 1$, $r^* = 2$, *and each rectilinear generator of the variety* X *bears a double focus* F_L^* *describing an* r*-dimensional variety* G *in the space* $\mathbb{P}^N$*, then* G *foliates into two-dimensional surfaces* G^2*, and each* G^2 *bears a one-parameter family of asymptotic lines. The variety* X *itself foliates into three-dimensional varieties* X^* *with degenerate Gauss maps of rank two formed by the tangents to the asymptotic lines of the surfaces* G^2.

(e) *If* $l = 1$, $r^* = 2$, *and the rectilinear generators of the variety* X *bears a double focus* F_L^* *describing an* $(r-1)$*-dimensional variety* G *in the space* $\mathbb{P}^N$*, then the variety* X *foliates into an* $(r-2)$*-parameter family of twisted cones of rank two formed by pencils of straight lines in the planes tangent to the curves* $C \subset G$.

Most likely, statements (d) and (e) can be generalized for the cases when $r^* > 2$.

Thus, Theorems 4.16 and 4.17 describe the structure of general varieties with degenerate Gauss maps. As a result, these theorems are *structure theorems* for such varieties.

Note that Theorem 4.17 does not cover varieties with degenerate Gauss maps with multiple nonlinear components of their focal images.

This gives rise to the following problem (see Akivis and Goldberg [AG 01a]):

Problem. *Construct an example of a variety* $X \subset \mathbb{P}^N(\mathbb{C})$ *with a degenerate Gauss map whose focal images have multiple nonlinear components or prove that such varieties do not exist. It is assumed that the variety* X *itself does not have multiple components.*

4.6 Embedding Theorems for Varieties with Degenerate Gauss Maps

4.6.1 The Embedding Theorem. In this section we prove the theorem for varieties X with degenerate Gauss maps giving sufficient conditions for X to be embedded into a subspace $\mathbb{P}^M$ of the space $\mathbb{P}^N$, $M < N$. The dual theorem gives sufficient conditions for X to be a cone.

Theorem 4.18. *Let $X \subset \mathbb{P}^N$ be a variety with a degenerate Gauss map of dimension n and rank $r < n$. Suppose that all matrices B^α can be simultaneously diagonalized, $B^\alpha = \text{diag}\,(B^\alpha_{pp})$. Suppose also that the rectangular $(r \times (N-n))$-matrix $B = (b^\alpha_{pp})$ composed from the eigenvalues of the matrices B^α has a rank $r_1 \leq r-1$, and this rank is not reduced when we delete any column of this matrix. Then the variety X belongs to a subspace $\mathbb{P}^{n+r_1}$ of the space $\mathbb{P}^N$.*

Proof. Under the conditions of Theorem 4.18, equations (3.4) takes the form

$$\omega^\alpha_p = b^\alpha_{pp}\omega^p, \ \ p = l+1, \ldots, n, \ \alpha = n+1, \ldots, N. \tag{4.101}$$

The matrix B has only r_1 linearly independent rows. Thus by means of transformations of the moving frame vertices located outside of the tangent subspace T_L, equations (4.101) can be reduced to the form

$$\omega^\lambda_p = b^\lambda_{pp}\omega^p, \ \ \omega^\sigma_p = 0, \tag{4.102}$$

where $\lambda = n+1, \ldots, n+r_1, \ \sigma = n+r_1+1, \ldots, N$. The third of equations (3.5) takes the form

$$dA_p = \omega^0_p A_0 + \omega^a_p A_a + \omega^q_p A_q + \omega^\lambda_p A_\lambda,$$

and the points A_λ together with the points A_0, A_a, and A_q define the osculating subspace T^2_L of the variety X for all points $x \in L$. The dimension of T^2_L is $n+r_1$, $\dim T^2_L = n + r_1$.

Differentiation of the points A_λ gives

$$dA_\lambda \equiv \omega^\sigma_\lambda A_\rho \pmod{T^2_L}, \tag{4.103}$$

where $\lambda, \mu = n+1, \ldots, n+r_1$; $\sigma = n+r_1+1, \ldots, N$. If $\omega^p = 0$, then the osculating subspace T^2_L of X remains fixed. It follows from equations (4.103) that the 1-forms ω^σ_λ are expressed in terms of the basis forms ω^p of X, that is,

$$\omega^\sigma_\lambda = l^\sigma_{\lambda p}\omega^p. \tag{4.104}$$

Taking exterior derivatives of the second group of equations (4.102), we find that

$$\omega^\lambda_p \wedge \omega^\sigma_\lambda = 0. \tag{4.105}$$

Substituting the values of the 1-forms ω^λ_p and ω^σ_λ from equations (4.102) and (4.104) into equation (4.105), we find that

$$b^\lambda_{pp}\omega^p \wedge l^\sigma_{\lambda q}\omega^q = 0.$$

In this equation the summation is carried over the indices λ and q, but there is no summation over the index p. It follows from these equations that

$$b^\lambda_{pp} l^\sigma_{\lambda q} = 0, \;\; p \neq q. \tag{4.106}$$

System (4.106) is a system of linear homogeneous equations with respect to the unknown variables $l^\sigma_{\lambda q}$. For each pair of the values σ and q, system (4.106) has the rank $r-1$ and r_1 unknowns. Because $r_1 \leq r-1$, under the conditions of Theorem 4.18, the rank of the matrix of coefficients of this system is equal to r_1. As a result, the system has only the trivial solution $l^\sigma_{\lambda q} = 0$. Thus, equations (4.104) take the form

$$\omega^\lambda_\sigma = 0. \tag{4.107}$$

It follows from (4.103) and (4.107) that the osculating subspace T^2_L of X remains fixed when L moves in X. Thus $X \subset \mathbb{P}^{n+r_1}$. □

Remark. If $r_1 = r$ and $N > n + r$, then the osculating subspace T^2_L of X can move in $\mathbb{P}^N$ when L moves in X. In this case the variety X is torsal.

Theorem 4.18 generalizes Theorem 2.1 proved in Section 2.2. The latter is similar to Theorem 3.10 from the book [AG 93] by Akivis and Goldberg and was proved in [AG 93] for varieties of a space $\mathbb{P}^N$ bearing a net of conjugate lines. As we noted in Chapter 2, this theorem generalizes a similar theorem of C. Segre (see [SegC 07], p. 571) proved for varieties X of dimension n of the space $\mathbb{P}^N$, which has at each point $x \in X$ the osculating subspace T^2_x of dimension $n+1$. By this theorem, a variety X either belongs to a subspace $\mathbb{P}^{n+1}$ or is a torse.

4.6.2 A Sufficient Condition for a Variety with a Degenerate Gauss Map to be a Cone. The theorem dual to Theorem 4.18 is also valid and gives a sufficient condition for a variety with a degenerate Gauss map to be a cone.

Theorem 4.19. *Let $X \subset \mathbb{P}^N$ be an n-dimensional variety with a degenerate Gauss map of rank $r < n$. Suppose that all matrices C_a can be simultaneously diagonalized, $C_a = \operatorname{diag}(c^p_{ap})$. Suppose also that the rectangular $(r \times l)$-matrix $C = (c^p_{ap})$ composed from the eigenvalues of the matrices C_a has a rank $r_2 \leq r-1$, and this rank is not reduced when we delete any column of this matrix. Then the variety X is a cone with an $(l - r_2)$-dimensional vertex K_L.*

Proof. The proof of this theorem is similar to the proof of Theorem 4.18. □

NOTES

4.1–4.2. The basic types of varieties with degenerate Gauss maps (torsal varieties, hypersurfaces, and cones) were considered in the recent paper [AG 01a] by Akivis and Goldberg. Note that the hypersurfaces with degenerate Gauss maps as a basic type were omitted in the paper [GH 79] by Griffiths and Harris.

The results presented in Example 4.6 are due to Safaryan [Saf 70] (see also Section 4.6 of the book [AG 93] by Akivis and Goldberg).

4.3. Hartman and Nirenberg indicated in [HN 59]: "A similar result under weaker differentiability hypotheses has been stated for $n = 2$ by Pogorelov" (see [P 56a, 56b]). For $\mathbb{E}^3$, the results of [HN 59] were developed further by Stocker (see [Sto 61, 69]).

In recent papers (see, for example, Ishikawa's papers [I 99a, 99b]) the authors state the Hartman–Nirenberg cylinder theorem by saying that "*a properly embedded developable hypersurface in* $\mathbb{E}^{n+1}$ *of* $\operatorname{rank}(\gamma) \leq 1$ *is necessarily a cylinder*." A similar result is known for a complex Euclidean space $\mathbb{C}^{n+1}$ (see the paper by Abe [Ab 72]; see also the survey [Bor 97] by Borisenko).

Note that an affine cylinder theorem in other formulations was presented in the paper [NP 87] by Nomizu and Pinkall (see also the book [NS 94] by Nomizu and Sasaki) and in the papers [O 95, 96, 98] by Opozda. Their affine cylinder theorems give sufficient conditions for a hypersurface (i.e., a variety of codimension one) X in A^{n+1} to be a cylinder erected over a curve with $(n-1)$-dimensional plane generators. Our affine cylinder theorem (Theorem 4.10) gives sufficient conditions for a variety X of *any codimension and any rank* r, $2 \leq r \leq n-1$, in A^N, $N-n \geq 2$, to be a cylinder erected over a submanifold of dimension r and rank r with $(n-r)$-dimensional plane generators. In the recent papers [Pio 01, 02a, 02b], Piontkowski considered in $\mathbb{P}^N$ complete varieties with degenerate Gauss maps with rank equal to two, three, and four and with all singularities located at a hyperplane at infinity. In particular, as an extreme case, he obtained an affine cylinder theorem for varieties of rank one and any codimension. So our affine cylinder theorem for varieties of codimension greater than two and rank $r \geq 2$ complements substantially all previously known affine cylinder theorems that were for hypersurfaces of rank one.

4.4. In subsections 4.4.1–4.4.3 we follow the paper [AG 03b] by Akivis and Goldberg.

Recently Wu and F. Zheng [WZ 02] considered a variety X of dimension n with a degenerate Gauss map in the complex Euclidean space $\mathbb{C}^N$ and proved that if $r \leq 4$ or $r = n-1$, then X is a twisted cylinder, i.e., it is foliated by cylinders (which reduce to $(n-1)$-planes when $r = 2$) whose generators are level sets of the Gauss map. This was conjectured by Vitter [V 79] for any value of r (and proved by him for $r = 2$), but the authors give counterexamples showing that it fails to be true for $r = 5$.

As we indicated in the Notes to Chapter 3, a classification of three-dimensional varieties with degenerate Gauss maps was presented in the papers [Rog 97] by Rogora and [MT 02a] by Mezzetti and Tommasi. In particular, in these papers, the varieties of the class g) were described as bands (although Rogora did not use this term). The description of a variety of this class as a twisted cone appeared in this book for the first time.

4.5–4.6. In these sections we follow the paper [AG 01a] by Akivis and Goldberg.

The problem at the end of Section 4.5 was posed by Akivis and Goldberg in their paper [AG 01a]. In the recent preprint [MT 02c], Mezzetti and Tommasi constructed a series of examples of varieties that solve the problem.

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Chapter 5

Further Examples and Applications of the Theory of Varieties with Degenerate Gauss Maps

In Section 5.1, we define the de Sitter space, prove that lightlike hypersurfaces in such a space have degenerate Gauss maps, that their rank $r \leq n-1$, and that there are singular points and singular submanifolds on them. We classify singular points and describe the structure of lightlike hypersurfaces carrying singular points of different types. Moreover, we establish the connection of this classification with that of canal hypersurfaces of the conformal space. The cases of maximal rank $r = n-1$ and of reduced rank $r < n-1$ are considered separately. In Section 5.2, we establish a relation of the theory of varieties with degenerate Gauss maps in projective spaces with the theory of congruences and pseudocongruences of subspaces and show how these two theories can be applied to the construction of induced connections on submanifolds of projective spaces and other spaces endowed with a projective structure. In Section 5.3, we consider smooth lines on projective planes over the complete matrix algebra $\mathbb{M}$ of order two, the algebra $\mathbb{C}$ of complex numbers, the algebra $\mathbb{C}^1$ of double numbers, and the algebra $\mathbb{C}^0$ of dual numbers. For the algebras $\mathbb{C}, \mathbb{C}^1$, and $\mathbb{C}^0$, in the space $\mathbb{RP}^5$, to these smooth lines there correspond families of straight lines describing three-dimensional point varieties X^3 with degenerate Gauss maps of rank $r \leq 2$. We prove that they represent examples of different types of varieties X^3 with degenerate Gauss maps. Namely, the variety X^3, corresponding in $\mathbb{RP}^5$ to a smooth line $\Gamma \subset \mathbb{CP}^2$, does not have real singular points, the variety X^3, corresponding in $\mathbb{RP}^5$ to a smooth line $\Gamma \subset \mathbb{C}^1\mathbb{P}^2$, bears two plane singular lines, and finally the variety X^3, corresponding in $\mathbb{RP}^5$ to a smooth line $\Gamma \subset \mathbb{C}^0\mathbb{P}^2$, bears one double singular line. In the last case, the variety X^3 is a generalization of Sacksteder–Bourgain's hypersurfaces with degenerate Gauss maps without singularities in the Euclidean space $\mathbb{E}^4$. We define the notion of the curvature of a smooth line in the plane $\mathbb{AP}^2$, $\mathbb{A} = \mathbb{C}, \mathbb{C}^1, \mathbb{C}^0$, and we prove that in all three cases, the rank of X^3 equals the rank of the curvature of the line Γ.

5.1 Lightlike Hypersurfaces in the de Sitter Space and Their Focal Properties

5.1.1 Lightlike Hypersurfaces and Physics. It is well known that the pseudo-Riemannian manifolds (M, g) of Lorentzian signature play a special role in geometry and physics and that they are models of spacetime of general relativity. At the tangent space T_x of an arbitrary point x of such a manifold, one can invariantly define a real isotropic cone C_x. From the point of view of physics, this cone is the light cone: trajectories of light impulses emanating from the point x are tangent to this cone.

An $(n+1)$-dimensional Riemannian manifold(M, g) is called *Lorentzian* if its metric tensor g has the signature $(n, 1)$. Hypersurfaces of a Lorentzian manifold (M, g) can be of three types: spacelike, timelike, and lightlike (see, for example, the books [ON 83] by O'Neill or [AG 96] by Akivis and Goldberg).

The tangent hyperplane to a spacelike hypersurface $X, \dim X = n$, of a Lorentzian manifold (M, g) at any point does not have real common points with the light cone C_x. This implies that on X a proper Riemannian metric is induced. The tangent hyperplane to a timelike hypersurface X at any point intersects the light cone C_x along an $(n-1)$-dimensional cone. This implies that on X a pseudo-Riemannian metric of Lorentzian signature $(n-1, 1)$ is induced. Finally, the tangent hyperplane to a lightlike hypersurface X at any point is tangent to the light cones C_x. This implies that on X a degenerate Riemannian metric signature $(n-1, 0)$ is induced.

A Lorentzian manifold of constant positive curvature is called the *de Sitter space*. The de Sitter space $\mathbb{S}_1^{n+1}$ admits a realization on the exterior of an n-dimensional oval hyperquadric Q^n of a projective space $\mathbb{P}^{n+1}$. Thus the de Sitter space is isometric to a pseudoelliptic space, $\mathbb{S}_1^{n+1} \sim \text{ext } Q^n$. Because the interior of the hyperquadric Q^n is isometric to the hyperbolic geometry of the Lobachevsky space $\mathbb{H}^{n+1}$, $\mathbb{H}^{n+1} \sim \text{int } Q^n$ and the geometry of Q^n itself is equivalent to that of an n-dimensional conformal space C^n, $C^n \sim Q^n$, the groups of motions of these three spaces are isomorphic to each other and are isomorphic to the group $\mathbf{SO}(n+2, 1)$ of rotations of a pseudo-Euclidean space $\mathbb{E}_1^{n+2}$ of Lorentzian signature. This allows us to apply the apparatus developed in the book [AG 96] by Akivis and Goldberg for the conformal space C^n to the study of the de Sitter space.

As we will show in this section, lightlike varieties in the de Sitter space are varieties with degenerate Gauss maps. For this reason we study them in this book. From the point of view of physics, lightlike hypersurfaces are of great importance because they are models of different types of horizons studied in general relativity: event horizons, Cauchy's horizons, Kruskal's horizons (see the books [Ch 83] by Chandrasekhar and [MTW 73] by Misner, Thorpe, and Wheeler). This is why the study of geometric structure of lightlike hypersurfaces is of interest.

5.1.2 The de Sitter Space. In a projective space $\mathbb{P}^{n+1}$ of dimension $n+1$, we consider an oval hyperquadric Q^n. Let x be a point of the space $\mathbb{P}^{n+1}$ with projective coordinates $(x^0, x^1, \ldots, x^{n+1})$. The hyperquadric Q^n is determined by the equations

$$(x, x) := g_{\xi\eta} x^\xi x^\eta = 0, \qquad \xi, \eta = 0, \ldots, n+1, \tag{5.1}$$

whose left-hand side is a quadratic form (x, x) of signature $(n+1, 1)$. The hyperquadric Q^n divides the space $\mathbb{P}^{n+1}$ into two parts, external and internal. We normalize the quadratic form (x, x) in such a way that for the points of the external part the inequality $(x, x) > 0$ holds. This external domain is a model of the *de Sitter space* $\mathbb{S}_1^{n+1}$ (see, for example, Y. Zheng [Z 96]). We identify the external domain of Q^n with the space $\mathbb{S}_1^{n+1}$. The hyperquadric Q^n is the *absolute* of the space $\mathbb{S}_1^{n+1}$.

On the hyperquadric Q^n of the space $\mathbb{P}^{n+1}$, the geometry of a conformal space C^n is realized. The bijective mapping $C^n \leftrightarrow Q^n$ is called the *Darboux mapping*, and the hyperquadric Q^n itself is called the *Darboux hyperquadric.*

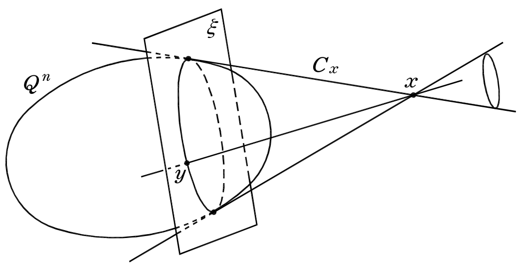

Figure 5.1

The Darboux mapping sends hyperspheres of the space C^n to cross sections of the hyperquadric Q^n by hyperplanes ξ. It also sends a hyperplane ξ to a point x that is polar-conjugate to ξ with respect to Q^n and lies outside of Q^n, that is, a point of the space $\mathbb{S}_1^{n+1}$ (see Figure 5.1). Thus, points of the space $\mathbb{S}_1^{n+1}$ correspond to hyperspheres of the space C^n.

Let x be an arbitrary point of the space $\mathbb{S}_1^{n+1}$. The tangent lines from the point x to the hyperquadric Q^n form a second-order cone C_x with vertex at the point x. This cone is called the *isotropic cone.* For spacetime, whose model is the space $\mathbb{S}_1^{n+1}$, this cone is the *light cone*, and its generators are lines of propagation of light impulses whose source coincides with point x.

The cone C_x divides all straight lines passing through the point x into *spacelike* (not having common points with the hyperquadric Q^n), *timelike* (intersecting Q^n in two different points), and *lightlike* (tangent to Q^n) straight lines. The lightlike straight lines are generators of the cone C_x.

A spacelike straight line $l \subset \mathbb{S}_1^{n+1}$ corresponds to an elliptic pencil of hyperspheres in the conformal space C^n. All hyperspheres of this pencil pass through a common $(n-2)$-sphere S^{n-2} (the center of this pencil). The sphere S^{n-2} is the intersection of the hyperquadric Q^n and the $(n-1)$-dimensional subspace of the space $\mathbb{P}^{n+1}$ that is polar-conjugate to the line l with respect to the hyperquadric Q^n.

A timelike straight line $l \subset \mathbb{S}_1^{n+1}$ corresponds to a hyperbolic pencil of hyperspheres in the space C^n. Two arbitrary hyperspheres of this pencil do not have common points, and the pencil contains two hyperspheres of zero radius that correspond to the points of intersection of the straight line l and the hyperquadric Q^n.

Finally, a lightlike straight line $l \subset \mathbb{S}_1^{n+1}$ corresponds to a parabolic pencil of hyperspheres in the space C^n consisting of hyperspheres tangent to each other at a point that is a unique hypersphere of zero radius belonging to this pencil.

Hyperplanes of the space $\mathbb{S}_1^{n+1}$ are also divided into three types. Spacelike hyperplanes do not have common points with the hyperquadric Q^n; a timelike hyperplane intersects Q^n along a real hypersphere; and lightlike hyperplanes are tangent to Q^n. Subspaces of any dimension r, $2 \leq r \leq n-1$, can be classified in a similar manner.

Let us apply the method of moving frames to study some questions of differential geometry of the space $\mathbb{S}_1^{n+1}$. With a point $x \in \mathbb{S}_1^{n+1}$ we associate a family of projective frames $\{A_0, A_1, \ldots, A_{n+1}\}$. In order to apply formulas derived in the book [AG 96] by Akivis and Goldberg, we will use the notations used in that book. Namely, we denote by A_n the vertex of the moving frame that coincides with the point x, $A_n = x$; we locate the vertices A_0, A_i $i = 1, \ldots, n-1,$, and A_{n+1} at the hyperplane ξ that is polar-conjugate to the point x with respect to the hyperquadric Q^n, and we assume that the points A_0 and A_{n+1} lie on the hypersphere $S^{n-1} = Q^n \cap \xi$, and the points A_i are polar-conjugate to the straight line $A_0 \wedge A_{n+1}$ with respect to S^{n-1}. Because $(x, x) > 0$, we can normalize the point A_n by the condition $(A_n, A_n) = 1$. The points A_0 and A_{n+1} are not polar-conjugate with respect to the hyperquadric Q^n. Hence we can normalize them by the condition $(A_0, A_{n+1}) = -1$. As a result, the matrix of scalar products of the frame elements has the form

$$(A_\xi, A_\eta) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & g_{ij} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad i, j = 1, \dots, n-1, \tag{5.2}$$

and the quadratic form (x, x) takes the form

$$(x, x) = g_{ij} x^i x^j + (x^n)^2 - 2x^0 x^{n+1}. \tag{5.3}$$

The quadratic form $g_{ij} x^i x^j$ occurring in (5.3) is positive definite.

The equations of infinitesimal displacement of the conformal frame $\{A_\xi\}$, $\xi = 0, 1, \dots, n+1$, we have constructed, have the form

$$dA_\xi = \omega_\xi^\eta A_\eta, \quad \xi, \eta = 0, 1, \dots, n+1, \tag{5.4}$$

where by (5.2), the 1-forms ω_ξ^η satisfy the following Pfaffian equations:

$$\begin{cases} \omega_0^{n+1} = \omega_{n+1}^0 = 0, & \omega_0^0 + \omega_{n+1}^{n+1} = 0, \\ \omega_i^{n+1} = g_{ij}\omega_0^j, & \omega_i^0 = g_{ij}\omega_{n+1}^j, \\ \omega_n^{n+1} - \omega_0^n = 0, & \omega_n^0 - \omega_{n+1}^n = 0, \\ g_{ij}\omega_n^j + \omega_i^n = 0, & \omega_n^n = 0, \\ dg_{ij} = g_{jk}\omega_i^k + g_{ik}\omega_j^k. & \end{cases} \tag{5.5}$$

These formulas are precisely the formulas derived in the book [AG 96] (see p. 32) by Akivis and Goldberg for the conformal space C^n.

It follows from (5.4) that

$$dA_n = \omega_n^0 A_0 + \omega_n^i A_i + \omega_n^{n+1} A_{n+1}. \tag{5.6}$$

The differential dA_n belongs to the tangent space $T_x(\mathbb{S}_1^{n+1})$, and the 1-forms ω_n^0, ω_n^i, and ω_n^{n+1} form a coframe of this space. The total number of these forms is $n+1$, and this number coincides with the dimension of $T_x(\mathbb{S}_1^{n+1})$. The scalar square of the differential dA_n is the metric quadratic form $\widetilde{g}$ on the manifold $\mathbb{S}_1^{n+1}$. By (5.2), this quadratic form $\widetilde{g}$ can be written as

$$\widetilde{g} = (dA_n, dA_n) = g_{ij}\omega_n^i \omega_n^j - 2\omega_n^0 \omega_n^{n+1}.$$

Because the first term of this expression is a positive definite quadratic form, the form $\widetilde{g}$ is of Lorentzian signature $(n, 1)$. The coefficients of the form $\widetilde{g}$

produce the metric tensor of the space $\mathbb{S}^{n+1}_1$ whose matrix is obtained from the matrix (5.2) by deleting the nth row and the nth column.

The quadratic form $\widetilde{g}$ defines on $\mathbb{S}^{n+1}_1$ a pseudo-Riemannian metric of signature $(n, 1)$. The isotropic cone defined in the space $T_x(\mathbb{S}^{n+1}_1)$ by the equation $\widetilde{g} = 0$ coincides with the cone C_x that we defined earlier in the space $\mathbb{S}^{n+1}_1$ geometrically.

The 1-forms ω^η_ξ occurring in equations (5.4) satisfy the structure equations of the space C^n:

$$d\omega^\eta_\xi = \omega^\zeta_\xi \wedge \omega^\eta_\zeta, \tag{5.7}$$

which are obtained by taking exterior derivatives of equations (5.4) and which are conditions of complete integrability of (5.4). The forms ω^η_ξ are invariant forms of the fundamental group $\mathbf{PO}(n+2, 1)$ of transformations of the spaces $\mathbb{H}^{n+1}, C^n$, and $\mathbb{S}^{n+1}_1$ which is locally isomorphic to the group $\mathbf{SO}(n+2, 1)$.

Let us write equations (5.7) for the 1-forms ω^0_n, ω^i_n, and ω^{n+1}_n making up a coframe of the space $T_x(\mathbb{S}^{n+1}_1)$ in more detail:

$$\left\{\begin{array}{ll} d\omega^0_n &= \omega^0_n \wedge \omega^0_0 + \omega^i_n \wedge \omega^0_i, \\ d\omega^i_n &= \omega^0_n \wedge \omega^i_0 + \omega^j_n \wedge \omega^i_j + \omega^{n+1}_n \wedge \omega^i_{n+1}, \\ d\omega^{n+1}_n &= \omega^i_n \wedge \omega^{n+1}_i + \omega^{n+1}_n \wedge \omega^{n+1}_{n+1}. \end{array}\right. \tag{5.8}$$

The last equations can be written in the matrix form as follows:

$$d\theta = -\omega \wedge \theta, \tag{5.9}$$

where $\theta = (\omega^u_n), u = 0, i, n{+}1$, is the column matrix with its values in the vector space $T_x(\mathbb{S}^{n+1}_1)$, and $\omega = (\omega^u_v), u, v = 0, i, n+1$, is a square matrix of order $n+1$ with values in the Lie algebra of the group of admissible transformations of coframes of the space $T_x(\mathbb{S}^{n+1}_1)$. The form ω is the connection form of the space $\mathbb{S}^{n+1}_1$. In detail this form can be written as

$$\omega = \begin{pmatrix} \omega^0_0 & \omega^0_i & 0 \\ \omega^i_0 & \omega^j_i & \omega^i_{n+1} \\ 0 & \omega^{n+1}_i & \omega^{n+1}_{n+1} \end{pmatrix}. \tag{5.10}$$

By (5.5), in this matrix, only the forms in the upper-left corner, which form an $n \times n$-matrix, are linearly independent.

Next we find the curvature form and the curvature tensor of the space $\mathbb{S}^{n+1}_1$. To this end, we take exterior derivative of the connection form ω, more precisely, of its independent part. Applying equations (5.7), we find the following components of the curvature form:

$$\begin{cases} \Omega_0^0 = d\omega_0^0 - \omega_0^i \wedge \omega_i^0 = \omega_n^{n+1} \wedge \omega_n^0, \\ \Omega_0^i = d\omega_0^i - \omega_0^0 \wedge \omega_0^i - \omega_0^j \wedge \omega_j^i = \omega_n^{n+1} \wedge \omega_n^i, \\ \Omega_i^0 = d\omega_i^0 - \omega_i^0 \wedge \omega_0^0 - \omega_i^j \wedge \omega_j^0 = -g_{ij}\omega_n^j \wedge \omega_n^0, \\ \Omega_j^i = d\omega_j^i - \omega_j^0 \wedge \omega_0^i - \omega_j^k \wedge \omega_k^i - \omega_j^{n+1} \wedge \omega_{n+1}^i = -g_{jk}\omega_n^k \wedge \omega_n^i. \end{cases} \tag{5.11}$$

But the general expression of the curvature form of an $(n+1)$-dimensional pseudo-Riemannian space with a coframe ω_n^0, ω_n^i, and ω_n^{n+1} is

$$\Omega_s^r = d\omega_s^r - \omega_s^t \wedge \omega_t^r = \frac{1}{2} R_{suv}^r \omega_n^u \wedge \omega_n^v, \tag{5.12}$$

where $r, s, t, u, v = 0, 1, \ldots, n-1, n+1$ (see, for example, Wolf [W 77]). Comparing equations (5.11) and (5.12), we find that

$$\Omega_s^r = \omega_n^r \wedge g_{sv}\omega_n^v. \tag{5.13}$$

It follows from (5.13) that

$$R_{suv}^r = \delta_u^r g_{sv} - \delta_v^r g_{su}, \tag{5.14}$$

where (g_{sv}) is the matrix of coefficients of the quadratic form (5.3). But this means that the space $\mathbb{S}_1^{n+1}$ is a pseudo-Riemannian space of constant positive curvature $K = 1$. The Ricci tensor of this space has the form

$$R_{sv} = R_{srv}^r = n g_{sv}. \tag{5.15}$$

This confirms that the space $\mathbb{S}_1^{n+1}$, as any pseudo-Riemannian space of constant curvature, is the Einstein space.

Thus by means of the method of moving frame we proved the following well-known theorem (see, for example, Wolf [W 77]).

Theorem 5.1. *The de Sitter space, whose model is the domain of a projective space $\mathbb{P}^{n+1}$ lying outside of an oval hyperquadric Q^n, is a pseudo-Riemannian space of Lorentzian signature $(n, 1)$ and of constant positive curvature $K = 1$. This space is homogeneous, and its fundamental group* $\mathbf{PO}(n+2, 1)$ *is locally isomorphic to the special orthogonal group* $\mathbf{SO}(n+2, 1)$.

5.1.3 Lightlike Hypersurfaces in the de Sitter Space. A hypersurface X, $\dim X = n$, in the de Sitter space $\mathbb{S}_1^{n+1}$ is said to be *lightlike* if all its tangent hyperplanes are lightlike, that is, they are tangent to the hyperquadric Q^n, which is the absolute of the space $\mathbb{S}_1^{n+1}$.

Denote by x an arbitrary point of the hypersurface X, by η the tangent hyperplane to X at the point $x, \eta = T_x(X)$, and by y the point of tangency of the hyperplane η with the hyperquadric Q^n (see Figure 5.2). Next, as in Section 5.1.1, denote by ξ the hyperplane that is polar-conjugate to the point x with respect to the hyperquadric Q^n, and associate with a point x a family of projective frames such that $x = A_n$, $y = A_0$, the points A_i, $i = 1, \dots, n-1$, belong to the intersection of the hyperplanes ξ and η, $A_i \in \xi \cap \eta$, and the point A_{n+1}, as well as the point A_0, belong to the straight line that is polar-conjugate to the $(n-2)$-dimensional subspace spanned by the points A_i. In addition, we normalize the frame vertices in the same way as this was done in Section 5.1.2. Then the matrix of scalar products of the frame elements has the form (5.2), and the components of infinitesimal displacements of the moving frame satisfy the Pfaffian equations (5.5).

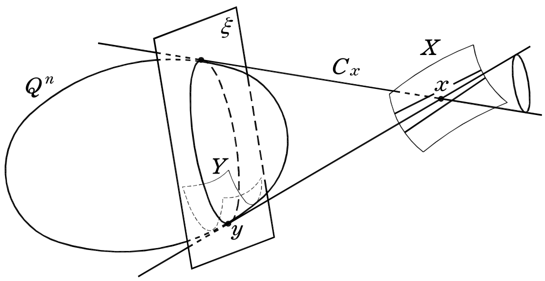

Figure 5.2

Because the hyperplane η is tangent to the hypersurface X at the point $x = A_n$ and does not contain the point A_{n+1}, the differential of the point $x = A_n$ has the form

$$dA_n = \omega_n^0 A_0 + \omega_n^i A_i, \tag{5.16}$$

the following equation holds:

$$\omega_n^{n+1} = 0, \tag{5.17}$$

and the forms ω_n^0 and ω_n^i are basis forms of the hypersurface X.

By relations (5.5), it follows from equation (5.17) that

$$\omega_0^n = 0 \tag{5.18}$$

and

$$dA_0 = \omega_0^0 A_0 + \omega_0^i A_i. \tag{5.19}$$

Taking the exterior derivative of equation (5.17), we obtain

$$\omega_n^i \wedge \omega_i^{n+1} = 0.$$

Because the forms ω_n^i are linearly independent, by Cartan's lemma, we find from the last equation that

$$\omega_i^{n+1} = \nu_{ij}\omega_n^j, \quad \nu_{ij} = \nu_{ji}. \tag{5.20}$$

Applying an appropriate formula from (5.5), we find that

$$\omega_0^i = g^{ij}\omega_j^{n+1} = g^{ik}\nu_{kj}\omega_n^j, \tag{5.21}$$

where (g^{ij}) is the inverse matrix of the matrix (g_{ij}).

Note that if $\omega_n^i = 0$, then by (5.20), $\omega_i^{n+1} = 0$, and by (5.5), $\omega_0^i = 0$. Now formulas (5.16) and (5.19) show that for $\omega_n^i = 0$, the isotropic straight line $A_n A_0$ is fixed, and hence X is a ruled hypersurface. In what follows, we assume that the *entire* straight line $A_n \wedge A_0$ belongs to the hypersurface X.

Thus the following theorem holds.

Theorem 5.2. *A lightlike hypersurface X of the de Sitter space $\mathbb{S}_1^{n+1}$ bears an $(n-1)$-parameter family of rectilinear generators $l = A_n \wedge A_0 \subset \mathbb{S}_1^{n+1}$ that are tangent to the absolute Q of this space at the points A_0.*

The rectilinear generators $l = A_n \wedge A_0$ forms on the hypersurface X a foliation (not a fibration), because, as we will show, each of them carries $r \leq n-1$ singular points (if each is counted as many times as its multiplicity), and this foliation is not locally trivial.

Next, we prove the following theorem.

Theorem 5.3. *A lightlike hypersurface X of the de Sitter space $\mathbb{S}_1^{n+1}$ is a ruled hypersurface with a degenerate Gauss map $\gamma : X \to \mathbb{G}(n, n+1)$ whose rank is equal to the dimension of the variety Y described by the point A_0 on the hyperquadric Q^n.*

Proof. Formulas (5.16) and (5.19) show that at any point of a generator of the hypersurface X, its tangent hyperplane is fixed and coincides with the hyperplane η. Thus X is a hypersurface with a degenerate Gauss map.

From relations (5.16) and (5.19) it follows that the tangent hyperplane η of the hypersurface X along its generator $A_n A_0$ is determined by this generator and the points A_i,

$$\eta = A_n \wedge A_0 \wedge A_1 \wedge \ldots \wedge A_{n-1}.$$

The displacement of this hyperplane is determined by the differentials (5.16), (5.19), and

$$dA_i = \omega_i^0 A_0 + \omega_i^j A_j + \omega_i^n A_n + \omega_i^{n+1} A_{n+1}.$$

But by (5.5), $\omega_i^n = -g_{ij}\omega_n^j$, and the forms ω_i^{n+1} are expressed according to formulas (5.50). From formulas (5.20) and (5.21) it follows that the rank of the hypersurface X is determined by the rank of the matrix (ν_{ij}) in terms of which the 1-forms ω_i^{n+1} and ω_0^i are expressed. But by (5.19) and (5.21), the dimension of the variety Y described by the point A_0 on the hyperquadric Q^n is also equal to the rank of the matrix (ν_{ij}). □

Denote the rank of the tensor ν_{ij}, which is equal to the rank of the hypersurface X, by r. In this and following sections we will assume that $r = n-1$, and the case $r < n-1$ will be considered in Section 5.1.4.

For $r = n-1$, the hypersurface X carries an $(n-1)$-parameter family of isotropic rectilinear generators $L = A_n \wedge A_0$ along which the tangent hyperplane $T_x(X)$ is fixed. From the point of view of physics, the isotropic rectilinear generators of a lightlike hypersurface X are trajectories of light impulses, and the hypersurface X itself represents a *light flux* in spacetime.

Because rank $(\nu_{ij}) = n-1$, the variety Y described by the point A_0 on the hyperquadric Q^n has dimension $n-1$, that is, Y is a hypersurface in Q^n, $\dim Y = n-1$. The tangent subspace $T_{A_0}(Y)$ to Y is determined by the points $A_0, A_1, \ldots, A_{n-1}$. Because $(A_n, A_i) = 0$, this tangent subspace is polar-conjugate to the rectilinear generator $A_n \wedge A_0$ of the lightlike hypersurface X.

The variety Y of the hyperquadric Q^n is the image of a hypersurface of the conformal space C^n under the Darboux mapping. We will denote this hypersurface also by Y. In the space C^n, the hypersurface Y is defined by equation (5.18) which by (5.5) is equivalent to equation (5.17) defining a lightlike hypersurface X in the space $\mathbb{S}_1^{n+1}$. To the rectilinear generator $A_n \wedge A_0$ of the hypersurface X, there corresponds a parabolic pencil of hyperspheres $A_n + sA_0$ tangent to the hypersurface Y (see the book [AG 96] by Akivis and Goldberg, p. 40). Thus the following theorem is valid.

Theorem 5.4. *There exists a one-to-one correspondence between the set of hypersurfaces of the conformal space C^n and the set of lightlike hypersurfaces of the maximal rank $r = n-1$ of the de Sitter space $\mathbb{S}_1^{n+1}$. To the pencils of tangent hyperspheres of the hypersurface Y there correspond the isotropic rectilinear generators of the lightlike hypersurface X.*

Note that for lightlike hypersurfaces of the four-dimensional Minkowski space M^4 the result similar to the result of Theorem 5.4 was obtained by Kossowski in [Kos 89].

5.1.4 Singular Points of Lightlike Hypersurfaces in the de Sitter Space. Suppose that the hypersurface X has the maximal rank $r = n-1$. This implies that X bears rectilinear generators $L = A_n \wedge A_0$. Taking exterior derivative of equations (5.18) defining the hypersurface Y in the conformal space C^n, we obtain

$$\omega_0^i \wedge \omega_i^n = 0,$$

from which by linear independence of the 1-forms ω_0^i on Y and Cartan's lemma we find that

$$\omega_i^n = b_{ij}\omega_0^j, \quad b_{ij} = b_{ji}. \tag{5.22}$$

It is not difficult to find relations between the coefficients ν_{ij} in formulas (5.20) and b_{ij} in formulas (5.22). Substituting the values of the forms ω_i^n and ω_0^j from (5.5) into (5.22), we find that

$$-g_{ij}\omega_n^j = b_{ij}g^{jk}\omega_k^{n+1}.$$

Solving these equations for ω_k^{n+1}, we obtain

$$\omega_i^{n+1} = -g_{ik}b^{kl}g_{lj}\omega_n^j,$$

where (b^{kl}) is the inverse matrix of the matrix (b_{ij}). Comparing these equations with equations (5.20), we obtain

$$\nu_{ij} = -g_{ik}b^{kl}g_{lj}. \tag{5.23}$$

Of course, in this computation we assumed that the matrix (b_{ij}) is nondegenerate.

Consider the point

$$z = A_n + sA_0 \tag{5.24}$$

on the rectilinear generator $L = A_n \wedge A_0$ of the hypersurface X. Differentiating this point and applying formulas (5.16) and (5.19), we obtain

$$dz = (ds + s\omega_0^0 + \omega_n^0)A_0 + (\omega_n^i + s\omega_0^i)A_i. \tag{5.25}$$

By (5.5) and (5.22), we have

$$\omega_n^i = g^{ik}\omega_k^n = -g^{ik}b_{kj}\omega_0^j.$$

As a result, formula (5.25) becomes

$$dz = (ds + s\omega_0^0 + \omega_n^0)A_0 - g^{ik}(b_{kj} - sg_{kj})\omega_0^j A_i. \tag{5.26}$$

The differential dz is the differential of the Gauss map $\gamma : X \to \mathbb{G}(n, n+1)$ that was considered in Theorem 5.2. The linearly independent forms ω_0^i are basis forms on the parametric manifold M^{n-1}, and the form $\theta = ds + s\omega_0^0 + \omega_n^0$ is a basis form on the line l. Thus the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & \hat{b}_j^i \end{pmatrix},$$

where $\hat{b}^i_j = g^{ik}(b_{kj} - sg_{kj})$, is the Jacobi matrix of this mapping.

By (5.26), the tangent subspace $T_z(X)$ to the hypersurface X at the point z is determined by the points z, A_0, and $\widehat{b}^i_j A_i$. Thus the point z is a *regular point* of the hypersurface X if and only if

$$\det(b_{ij} - sg_{ij}) \neq 0.$$

In particular, the condition $\det(b_{ij}) \neq 0$ is necessary and sufficient for the point $A_n \in L = A_n \wedge A_0$ to be regular.

The equation

$$\det(b_{ij} - sg_{ij}) = 0 \tag{5.27}$$

defines *singular points* on the rectilinear generator L of the hypersurface X. This equation is the *characteristic equation* of the matrix (b_{ij}) with respect to the matrix (g_{ij}). Because the matrices (b_{ij}) and (g_{ij}) are both symmetric, and (g_{ij}) defines the positive definite quadratic form g and is of rank $n-1$, equation (5.27) has $n-1$ real roots if each is counted as many times as its multiplicity. Thus each rectilinear generator $A_n \wedge A_0$ of a lightlike hypersurface X carries $n-1$ real singular points.

From the point of view of geometric optics, the singular points are the points of condensation of light rays on a lightlike hypersurface X, i.e., they are foci, and the varieties defined by them in the space $\mathbb{S}^{n+1}_1$ are the *focal varieties*, or the *caustics*, on X (cf. Example 3.5 on p. 103).

Denote the roots of the characteristic equation (5.27) by $s_h, h = 1, 2, \ldots, n-1$. Then the foci on the rectilinear generator $A_n \wedge A_0$ corresponding to these roots can be written in the form

$$F_h = A_n + s_h A_0. \tag{5.28}$$

It is clear from (5.28) that the point A_0 is not a focus of the rectilinear generator $A_n \wedge A_0$. This is explained by the fact that by our assumption rank $(\nu_{ij}) = n-1$, and by (5.21), on the hyperquadric Q^n the point A_0 describes a hypersurface Y that is transversal to the straight lines $A_n \wedge A_0$.

In the conformal theory of hypersurfaces, to the singular points F_h, there correspond the tangent hyperspheres defining the principal directions at a point A_0 of the hypersurface Y of the conformal space C^n (see p. 55 in the book [AG 96] by Akivis and Goldberg).

We now construct a classification of singular points of a lightlike hypersurface X of the space $\mathbb{S}^{n+1}_1$. We will use some computations made in the paper Akivis and Goldberg [AG 98a] in which a classification of canal hypersurfaces was constructed.

Suppose first that $F_1 = A_n + s_1 A_0$ is a singular point defined by a simple root s_1 of characteristic equation (5.27), $s_1 \neq s_h, h = 2, \dots, n-1$. For this singular point we have

$$dF_1 = (ds_1 + s_1\omega_0^0 + \omega_n^0)A_0 - \hat{b}_j^i \omega_0^j A_i, \tag{5.29}$$

where

$$\hat{b}_j^i = g^{ik}(b_{kj} - s_1 g_{kj}) \tag{5.30}$$

is a degenerate symmetric affinor having a single null eigenvalue. The matrix of this affinor can be reduced to a quasidiagonal form

$$(\hat{b}_j^i) = \begin{pmatrix} 0 & 0 \\ 0 & \hat{b}_q^p \end{pmatrix}, \tag{5.31}$$

where $p, q = 2, \dots, n-1$, and $(\hat{b}_q^p)$ is a nondegenerate symmetric affinor. The matrices (g_{ij}) and $(b_{ij} - s_1 g_{ij})$ are reduced to the forms

$$\begin{pmatrix} 1 & 0 \\ 0 & g_{pq} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & \hat{b}_{pq} \end{pmatrix},$$

where $(\hat{b}_{pq}) = (b_{pq} - s_1 g_{pq})$ is a nondegenerate symmetric matrix.

Because the point F_1 is defined invariantly on the generator $A_n \wedge A_0$, it is fixed if $\omega_0^i = 0$. Thus it follows from (5.29) that

$$ds_1 + s_1\omega_0^0 + \omega_n^0 = s_{1i}\omega^i, \tag{5.32}$$

here and in what follows $\omega^i = \omega_0^i$. By (5.31) and (5.32), relation (5.29) takes the form

$$dF_1 = s_{11}\omega^1 A_0 + (s_{1p}A_0 - \hat{b}_p^q A_q)\omega^p. \tag{5.33}$$

Here the points $C_p = s_{1p}A_0 - \hat{b}_p^q A_q$ are linearly independent and belong to the tangent subspace $T_x(X)$.

Consider the variety $\mathcal{F}_1$ described by the singular point F_1 in the space $\mathbb{S}_1^{n+1}$. This variety is called the *focal variety* of the hypersurface X. Relation (5.33) shows that two cases are possible:

1) $s_{11} \neq 0$. In this case the variety $\mathcal{F}_1$ is of dimension $n-1$, and its tangent subspace at the point F_1 is determined by the points F_1, A_0, and C_p. This subspace contains the straight line $A_n \wedge A_0$ and intersects the hyperquadric Q^n. Thus this subspace, as well as the variety $\mathcal{F}_1$ itself, is timelike. For $\omega^p = 0$, the point F_1 describes a curve γ on the variety $\mathcal{F}_1$, which is tangent to the

straight line $F_1 \wedge A_0$ coinciding with the generator $A_n \wedge A_0$ of the hypersurface X. The curve γ is an isotropic curve on the variety $\mathcal{F}_1$. Thus on $\mathcal{F}_1$ there arises a fiber bundle of focal lines. The hypersurface X foliates into an $(n-2)$-parameter family of torses for which these lines are edges of regressions. The points F_1 are singular points of a kind called a *fold*.

If the characteristic equation (5.27) has distinct roots, then an isotropic rectilinear generator l of a lightlike hypersurface X carries $n-1$ distinct foci $F_h, h = 1, \ldots, n-1$. If for each of these foci the condition of type $s_{11} \neq 0$ holds, then each of them describes a focal variety $\mathcal{F}_h$ of dimension $n-1$ carrying a conjugate net. Curves of one family of this net are tangent to the straight lines l, and this family is isotropic. On the hypersurface Y of the space $C^n = Q^n$ described by the point A_0, the net of curvature lines corresponds to these conjugate nets.

2) $s_{11} = 0$. In this case, relation (5.33) takes the form

$$dF_1 = (s_{1p}A_0 - \hat{b}_p^q A_q)\omega^p, \tag{5.34}$$

and the focal variety $\mathcal{F}_1$ is of dimension $n-2$. Its tangent subspace at the point F_1 is determined by the points F_1 and C_p. An arbitrary point z of this subspace can be written in the form

$$z = z^n F_1 + z^p C_p = z^n(A_n + s_1 A_0) + z^p(s_{1p}A_0 - \hat{b}_p^q A_q).$$

Substituting the coordinates of this point into relation (5.3), we find that

$$(z, z) = g_{rs}\hat{b}_p^r \hat{b}_q^s z^p z^q + (z^n)^2 > 0.$$

It follows that the tangent subspace $T_{F_1}(\mathcal{F}_1)$ does not have common points with the hyperquadric Q^n, that is, it is spacelike. Because this takes place for any point $F_1 \in \mathcal{F}_1$, the focal variety $\mathcal{F}_1$ is spacelike.

For $\omega^p = 0$, the point F_1 is fixed. The subspace $T_{F_1}(\mathcal{F}_1)$ is fixed too. On the hyperquadric Q^n, the point A_0 describes a curve q that is polar-conjugate to $T_{F_1}(\mathcal{F}_1)$. Because $\dim T_{F_1}(\mathcal{F}_1) = n-2$, the curve q is a conic, along which the two-dimensional plane polar-conjugate to the subspace $T_{F_1}(\mathcal{F}_1)$ with respect to the hyperquadric Q^n intersects Q^n. Thus for $\omega^p = 0$, the rectilinear generator $A_n \wedge A_0$ of the hypersurface X describes a two-dimensional second-order cone with vertex at the point F_1 and the directrix q. Hence in the case under consideration a lightlike hypersurface X foliates into an $(n-2)$-parameter family of second-order cones whose vertices describe the $(n-2)$-dimensional focal variety $\mathcal{F}_1$, and the points F_1 are *conic* singular points of the hypersurface X.

The hypersurface Y of the conformal space C^n corresponding to such a lightlike hypersurface X is a *canal hypersurface* that envelops an $(n-2)$-parameter family of hyperspheres. Such a hypersurface carries a family of cyclic generators that depends on the same number of parameters. Such hypersurfaces were investigated in detail in the paper Akivis and Goldberg [AG 98a].

Further let F_1 be a singular point of multiplicity m, where $m \geq 2$, of a rectilinear generator $A_n \wedge A_0$ of a lightlike hypersurface X of the space $\mathbb{S}_1^{n+1}$ defined by an m-multiple root of characteristic equation (5.27). We assume that

$$s_1 = s_2 = \ldots = s_m := s_0, \; s_0 \neq s_p, \tag{5.35}$$

and that $a, b, c = 1, \ldots, m$ and $p, q, r = m+1, \ldots, n-1$. Then the matrices (g_{ij}) and (b_{ij}) can be simultaneously reduced to quasidiagonal forms

$$\begin{pmatrix} g_{ab} & 0 \\ 0 & g_{pq} \end{pmatrix} \text{ and } \begin{pmatrix} s_0 g_{ab} & 0 \\ 0 & b_{pq} \end{pmatrix}. \tag{5.36}$$

We also construct the matrix $(\hat{b}_{ij}) = (b_{ij} - s_0 g_{ij})$. Then

$$(\hat{b}_{ij}) = \begin{pmatrix} 0 & 0 \\ 0 & \hat{b}_{pq} \end{pmatrix}, \tag{5.37}$$

where $\hat{b}_{pq} = b_{pq} - s_0 g_{pq}$ is a nondegenerate matrix of order $n-m-1$.

By (5.37) and formulas (5.5) and (5.22) we have

$$\omega_a^n - s_0 \omega_a^{n+1} = 0, \tag{5.38}$$

$$\omega_p^n - s_0 \omega_p^{n+1} = \hat{b}_{pq} \omega^q. \tag{5.39}$$

Note that using (5.5), (5.22), (5.36), and (5.37), we find that

$$\begin{array}{lll} \omega_b^n = s_0 g_{bc} \omega^c, & \omega_p^n = b_{pq} \omega^q, & \omega_a^{n+1} = g_{ab} \omega^b, \\ \omega_p^{n+1} = g_{pq} \omega^q, & \omega_{n+1}^{n+1} = -\omega_0^0, & \omega_p^n - \omega_p^{n+1} = \hat{b}_{pq} \omega^q. \end{array}$$

Taking the exterior derivative of equation (5.38) and applying the above relations, we find that

$$\hat{b}_{pq} \omega_a^p \wedge \omega^q + g_{ab} \omega^b \wedge (ds_0 + s_0 \omega_0^0 + \omega_n^0) = 0. \tag{5.40}$$

It follows that the 1-form $ds_0 + s_0 \omega_0^0 + \omega_n^0$ can be expressed in terms of the basis forms. We write these expressions in the form

$$ds_0 + s_0 \omega_0^0 + \omega_n^0 = s_{0c} \omega^c + s_{0q} \omega^q. \tag{5.41}$$

Substituting decomposition (5.41) into equation (5.40), we find that

$$(\hat{b}_{pq}\omega_a^p + g_{ab}s_{0q}\omega^b) \wedge \omega^q + g_{ab}s_{0c}\omega^b \wedge \omega^c = 0. \tag{5.42}$$

The left-hand side of (5.42) does not have similar terms. Hence both terms are equal to 0. Equating to 0 the coefficients of the summands of the second term, we find that

$$g_{ab}s_{0c} = g_{ac}s_{0b}. \tag{5.43}$$

Contracting this equation with the matrix (g^{ab}) which is the inverse matrix of the matrix (g_{ab}), we obtain

$$ms_{0c} = s_{0c}.$$

Because $m \geq 2$, it follows that

$$s_{0c} = 0,$$

and relation (5.41) takes the form

$$ds_0 + s_0\omega_0^0 + \omega_n^0 = s_{0p}\omega^p. \tag{5.44}$$

For the singular point F_1 of multiplicity m of the generator $A_n \wedge A_0$ in question, equation (5.29) can be written in the form

$$dF_1 = (ds_0 + s_0\omega_0^0 + \omega_n^0)A_0 - \hat{b}_q^p\omega_0^q A_p.$$

Substituting decomposition (5.44) in the last equation, we find that

$$dF_1 = (s_{0p}A_0 - \hat{b}_p^q A_q)\omega_0^p. \tag{5.45}$$

This relation is similar to equation (5.34) with the only difference being that in (5.34) we had $p, q = 2, \ldots, n-1$, and in (5.45) we have $p, q = m+1, \ldots, n-1$. Thus the point F_1 describes a spacelike focal variety $\mathcal{F}_1$ of dimension $n-m-1$. For $\omega_0^p = 0$, the point F_1 is fixed, and the point A_0 describes an m-dimensional variety on the hyperquadric Q^n, which is a cross section of Q^n by an $(m+1)$-dimensional subspace that is polar-conjugate to the $(n-m-1)$-dimensional subspace tangent to the variety $\mathcal{F}_1$.

The point F_1 is a conic singular point of multiplicity m of a lightlike hypersurface X, and this hypersurface foliates into an $(n-m-1)$-parameter family of $(m+1)$-dimensional second-order cones circumscribed about the hyperquadric Q^n. The hypersurface Y of the conformal space C^n that corresponds to such a hypersurface X is an m-canal hypersurface (i.e., the envelope of an $(n-m-1)$-parameter family of hyperspheres), and it carries an m-dimensional spherical generators.

Note also the extreme case when the rectilinear generator $L = A_n \wedge A_0$ of a lightlike hypersurface X carries a single singular point of multiplicity $n-1$. It follows from our consideration of the cases $m \geq 2$ that this singular point is fixed, and the hypersurface X becomes a second-order hypercone with vertex at this singular point which is circumscribed about the hyperquadric Q^n. This hypercone is the isotropic cone of the space $\mathbb{S}_1^{n+1}$. The hypersurface Y of the conformal space C^n that corresponds to such a hypersurface X is a hypersphere of the space C^n.

The following theorem combines the results of this section.

Theorem 5.5. *A lightlike hypersurface X of maximal rank $r = n-1$ of the de Sitter space $\mathbb{S}_1^{n+1}$ possesses $n-1$ real singular points on each of its rectilinear generators $L = A_n \wedge A_0$ if each of these singular points is counted as many times as its multiplicity. The simple singular points can be of two kinds: a fold and conic. In the first case, the hypersurface X foliates into an $(n-2)$-parameter family of torses, and in the second case, it foliates into an $(n-2)$-parameter family of second-order cones. The vertices of these cones describe the $(n-2)$-dimensional spacelike variety in the space $\mathbb{S}_1^{n+1}$. All multiple singular points of a hypersurface X are conic. If a rectilinear generator of a hypersurface X carries a singular point of multiplicity m, $2 \leq m \leq n-1$, then the hypersurface X foliates into an $(n-m-1)$-parameter family of $(m+1)$-dimensional second-order cones. The vertices of these cones describe the $(n-m-1)$-dimensional spacelike variety in the space $\mathbb{S}_1^{n+1}$. The hypersurface Y of the conformal space C^n corresponding to a lightlike hypersurface X with singular points of multiplicity m is a canal hypersurface that envelops an $(n-m-1)$-parameter family of hyperspheres and has m-dimensional spherical generators.*

Because lightlike hypersurfaces X of the de Sitter space $\mathbb{S}_1^{n+1}$ represent a light flux (see Section 5.1.2), its focal varieties have the following physical meaning. If one of them is a lighting variety, then others are varieties of concentration of the light flux. Intensity of concentration depends on multiplicity of a focus describing this variety. Each of these focal varieties is a *caustic*, i.e., a locus of concentration of light rays.

In the extreme case, when an isotropic rectilinear generator $L = A_n \wedge A_0$ of a hypersurface X carries one $(n-1)$-multiple focus, the hypersurfaces X degenerates into the light cone generated by a point source of light. This cone represents a radiating light flux.

If each isotropic generator $L \subset X$ carries two foci F_1 and F_2 of multiplicities m_1 and m_2, $m_1 + m_2 = n-1$, $m_1 > 1$, $m_2 > 1$, then these foci describe spacelike varieties $\mathcal{F}_1$ and $\mathcal{F}_2$ of dimension $n - m_1 - 1$ and $n - m_2 - 1$, respectively. If one of these varieties is a lighting variety, then on the second one a light flux is concentrated.

5.1.5 Lightlike Hypersurfaces of Reduced Rank in the de Sitter Space. As we proved in Section 5.1.2, lightlike hypersurfaces of the de Sitter space $\mathbb{S}_1^{n+1}$ are ruled hypersurfaces with degenerate Gauss maps. However, in Section 5.1.4, we assumed that the rank of these hypersurfaces is maximal, that is, it is equal to $n-1$. In this section we consider lightlike hypersurfaces of reduced rank $r < n-1$.

We proved in Section 5.1.3 that the rank of a lightlike hypersurface X coincides with the rank of the matrix (ν_{ij}) defined by equation (5.20) as well as with the dimension of the variety V described by the point A_0 on the Darboux hyperquadric Q^n. As a result, to a lightlike hypersurface X of rank r there corresponds an r-dimensional variety Y, $\dim Y = r$, in the conformal space C^n.

The symmetric matrices (g_{ij}) and (ν_{ij}), the first of which is nondegenerate and positive definite and the second of which is of rank r, can be simultaneously reduced to quasidiagonal forms

$$(g_{ij}) = \begin{pmatrix} g_{ab} & 0 \\ 0 & g_{pq} \end{pmatrix} \;\; \text{and} \;\; (\nu_{ij}) = \begin{pmatrix} 0 & 0 \\ 0 & \nu_{pq} \end{pmatrix}, \tag{5.46}$$

where $a, b = 1, \ldots, n-r-1$; $p, q, s = n-r, \ldots, n-1$, $\nu_{pq} = \nu_{qp}$, and $\det(\nu_{pq}) \neq 0$. This implies that formulas (5.21) take the form

$$\omega_0^a = 0, \;\; \omega_0^p = g^{ps}\nu_{sq}\omega_n^q. \tag{5.47}$$

The second equation in system (5.47) shows that the 1-forms ω_0^p are linearly independent: they are basis forms on the variety Y, $\dim Y = r$, described by the point A_0 on the hyperquadric Q^n, on the lightlike hypersurface X of rank r, and also on a frame bundle associated with this hypersurface. The 1-forms occurring in equations (5.4) as linear combinations of the basis forms ω_0^p are principal forms, and the 1-forms that are not expressed in terms of the basis forms are fiber forms on the above mentioned frame bundle.

By (5.5), the second group of equations (5.47) is equivalent to the system of equations

$$\omega_p^n = b_{pq}^n \omega_0^q, \tag{5.48}$$

where $b_{pq}^n = -g_{ps}\widetilde{\nu}^{st}g_{tq}$, $(\widetilde{\nu}^{st})$ is the inverse matrix of the matrix (ν_{pq}), $b_{pq}^n = b_{qp}^n$, and $\det(b_{pq}^n) \neq 0$. Note that we can also obtain equations (5.48) by differentiation of equation (5.18) which holds on the hypersurface X.

Taking exterior derivatives of the first group of equations (5.47), we find that

$$\omega_0^p \wedge \omega_p^a = 0.$$

Applying Cartan's lemma to this system, we find that

$$\omega_p^a = b_{pq}^a \omega_0^q, \;\; b_{pq}^a = b_{qp}^a. \tag{5.49}$$

Note also that equations (5.5) and (5.46) imply that

$$g_{pq}\omega_a^q + g_{ab}\omega_p^b = 0.$$

By (5.49), it follows from the last equation that

$$\omega_a^p = -g_{ab}g^{pq}b_{qs}^b\omega_0^s. \tag{5.50}$$

Note also that the quantities b_{pq}^a and b_{pq}^n are determined in a second-order neighborhood of a rectilinear generator $L = A_n \wedge A_0$ of the hypersurface X.

Let us prove that in our frame an $(m+1)$-dimensional span L of the points A_0, A_a, and A_n is a plane generator of the hypersurface X. In fact, it follows from equations (5.4), (5.46), and (5.49) that in the case in question we have

$$\begin{cases} dA_0 = \omega_0^0 A_0 & +\omega_0^p A_p, \\ dA_a = \omega_a^0 A_0 + \omega_a^b A_b & +\omega_a^p A_p + \omega_a^n A_n, \\ dA_n = \omega_n^0 A_0 + \omega_n^a A_a & +\omega_n^p A_p. \end{cases} \tag{5.51}$$

If we fix the principal parameters in equations (5.51) (i.e., if we assume that $\omega_0^p = 0$), we obtain

$$\begin{cases} \delta A_0 = \pi_0^0 A_0, \\ \delta A_a = \pi_a^0 A_0 + \pi_a^b A_b + \pi_a^n A_n, \\ \delta A_n = \pi_n^0 A_0 + \pi_n^a A_a. \end{cases} \tag{5.52}$$

In the last equations δ is the symbol of differentiation with respect to the fiber parameters (i.e., for $\omega_0^p = 0$), and $\pi_\eta^\xi = \omega_\eta^\xi(\delta)$.

Equations (5.52) show that for $\omega_0^p = 0$, the point A_n of the hypersurface X moves in an $(m+1)$-dimensional domain belonging to the subspace $L = A_0 \wedge A_1 \wedge \ldots \wedge A_m \wedge A_n$ of the same dimension. Let us assume that the entire subspace L belongs to the hypersurface X, and that the point $A_n \in L$ moves freely in L. The subspace L is tangent to the hyperquadric Q^n at the point $A_0 \in Y$, and thus L is lightlike. Because the point A_0 describes an r-dimensional variety, the family of subspaces L depends on r parameters.

Equations (5.51) and (5.52) show that the basis 1-forms of the lightlike hypersurface X are divided into two classes: ω_n^p and ω_n^a. The forms ω_n^p are connected with the displacement of the lightlike $(m+1)$-plane L in the space $\mathbb{S}_1^{n+1}$, and the forms ω_n^a are connected with the displacement of the straight line $A_n \wedge A_0$ in this $(m+1)$-plane. Because (5.51) implies that for $\omega_n^p = 0$ the point A_0 remains fixed, the rectilinear generator $A_n \wedge A_0$ describes an m-dimensional bundle of straight lines with its center at the point A_0, and this bundle belongs to the fixed $(m+1)$-dimensional subspace L passing through this point.

Further consider an arbitrary point

$$z = z^0 A_0 + z^a A_a + z^n A_n \tag{5.53}$$

of the generator L of the lightlike hypersurface X. From formulas (5.51) it follows that the differential of any such point belongs to one and the same n-dimensional subspace $A_0 \wedge \ldots \wedge A_n$ tangent to the hypersurface X at the original point A_n. The latter means that the tangent subspace to the hypersurface X is not changed when the point z moves along the lightlike generator L of the hypersurface X. Thus, the hypersurface X is a hypersurface with a degenerate Gauss map of rank r.

As a result, we arrive at the following theorem making Theorem 5.3 more precise.

Theorem 5.6. *If the rank of the tensor ν_{ij} defined by relation* (5.20) *is equal to r, $r < n-1$, then a lightlike hypersurface X of the de Sitter space $\mathbb{S}_1^{n+1}$ is a ruled hypersurface with a degenerate Gauss map of rank r with $(m+1)$-dimensional lightlike generators, $m = n-r-1$, along which the tangent hyperplanes of X are constant. The points of tangency of lightlike generators with the hyperquadric Q^n form an r-dimensional variety Y,* $\dim Y = r$, *on Q^n.*

The last fact mentioned in Theorem 5.6 can also be treated in terms of quadratic hyperbands (see the book [AG 93] by Akivis and Goldberg, p. 256). By Theorem 5.6, the hypersurface X is the envelope of an r-parameter family of hyperplanes η tangent to the hyperquadric Q^n at the points of an r-dimensional smooth submanifold Y belonging to this hyperquadric. But this coincides precisely with the definition of the quadratic hyperband. Thus Theorem 5.6 can be complemented as follows.

Theorem 5.7. *A lightlike hypersurface X of rank r in the de Sitter space $\mathbb{S}_1^{n+1}$ is an r-dimensional quadratic hyperband with the support submanifold Y,* $\dim Y = r$, *belonging to the Darboux hyperquadric Q^n.*

Note also the extreme case when the rank of a lightlike hypersurface X is equal to 0. Then we have

$$\nu_{ij} = 0, \;\; \omega_0^i = 0.$$

The point A_0 is fixed on the hyperquadric Q^n, and the point A_n moves freely in the hyperplane η tangent to the hyperquadric Q^n at the point A_0. The lightlike hypersurface X degenerates into the hyperplane η tangent to the hyperquadric Q^n at the point A_0, and the quadratic hyperband associated with X is reduced to a degenerate 0-pair consisting of the point A_0 and the hyperplane η.

Let us also find singular points on a rectilinear generator L of a lightlike hypersurface X of rank r of the de Sitter space $\mathbb{S}_1^{n+1}$. To this end, we write the differential of a point $z \in L$ defined by equation (5.53). We will be interested only in the part of this differential that does not belong to the generator L. By (5.51), we obtain

$$dz \equiv (z^0\omega_0^p + z^a\omega_a^p + z^n\omega_n^p)A_p \pmod{L}.$$

By (5.48), (5.49), and (5.50), we find from the last relation that

$$dz \equiv N_q^p(z)\omega_0^q A_p \pmod{L},$$

where

$$N_q^p(z) = \delta_q^p z^0 - g_{ab}g^{ps}b_{sq}^b z^a - g^{ps}b_{sq}^n z^n. \tag{5.54}$$

At singular points of a generator L the dimension of the tangent subspace $T_x(X)$ to the hypersurface X is reduced. By (5.54), this is equivalent to the reduction of the rank of the matrix $N_q^p(z)$. Thus singular points of generator L can be found from the condition

$$\det N_q^p(z) = 0, \tag{5.55}$$

which defines an algebraic focus hypersurface $\mathcal{F}$ order r in the $(m+1)$-dimensional plane generator L. The left-hand side of equation (5.55) is the Jacobian of the Gauss map $\gamma : X \rightarrow \mathbb{G}(n, n+1)$, and the focal variety $\mathcal{F}$ is the locus of singular points of this map that are located in the plane generator L of the hypersurface X indicated in Theorem 5.2 on p. 183.

If the rank of a lightlike hypersurface X is maximal, that is, it is equal to $r = n-1$, then its determinant manifold $\mathcal{F}$ is a set of singular points of its rectilinear generator $A_n \wedge A_0$ determined by equation (5.28). On the other hand, if $r < n-1$, then singular points of the straight lines $A_n \wedge A_0$ lying in the generator L are also determined by equation (5.28), and they are the common points of these straight lines and the variety $\mathcal{F}$.

5.2 Induced Connections on Submanifolds

5.2.1 Congruences and Pseudocongruences in a Projective Space. The theory of congruences and pseudocongruences of subspaces of a projective space is closely related to the theory of varieties with degenerate Gauss maps.

In a projective space $\mathbb{P}^N$, we consider a family Y of its l-dimensional subspaces L, $\dim L = l$, which depends on $r = n - l$ parameters. We assume that not more than a finite number of subspaces L passes through every point

$x \in \mathbb{P}^n$. If we restrict ourselves by a small neighborhood of a subspace L, then we can assume that only one subspace $L \subset Y$ passes through a generic point $x \in L$. Such families of the space $\mathbb{P}^n$ are called the *congruences*.

In a three-dimensional space $\mathbb{P}^3$ as well as in three-dimensional spaces endowed with a projective structure (such as an affine, Euclidean, and non-Euclidean space), the theory of congruences was studied by many geometers. The extensive monographs on this subject were published (see, for example, the monograph [Fi 50] by Finikov).

The dual image for a congruence Y of l-dimensional subspaces in $\mathbb{P}^n$ is a *pseudocongruence* Y^* which is an r-parameter family of subspaces of dimension $r-1$. Every hyperplane $\xi \subset \mathbb{P}^n$ contains not more than a finite number of subspaces $L^* \subset Y^*$. However, if we consider an infinitesimally small neighborhood of the subspace L^* of the pseudocongruence Y^*, then there is only a single subspace L^* in the hyperplane ξ.

In this section, we shall establish a relation of the theory of varieties with degenerate Gauss maps in projective spaces with the theory of congruences and pseudocongruences of subspaces and show how these two theories can be applied to the construction of induced connections on submanifolds of projective spaces and other spaces endowed with a projective structure.

So, consider in $\mathbb{P}^n$ a congruence Y of l-dimensional subspaces L. We associate with its element L a family of projective frames $\{A_0, A_1, \ldots, A_n\}$ chosen in such a way that the points $A_0, A_1, \ldots, A_l$ are located in L, and the points $A_{l+1}, \ldots, A_n$ are located outside of L. The equations of infinitesimal displacement of such frames have the form

$$\begin{cases} dA_i = \omega_i^j A_j + \omega_i^p A_p, \\ dA_p = \omega_p^i A_i + \omega_p^q A_q, \end{cases} \tag{5.56}$$

where $i, j = 0, 1, \ldots, l$; $p, q = l+1, \ldots, n$, and $L = A_0 \wedge A_1 \ldots \wedge A_l$ is a generator of the congruence Y in question. Because this generator depends on r parameters and is fixed, when $\omega_i^p = 0$, the forms ω_i^p are expressed linearly in terms of the differentials of these r parameters or in terms of linearly independent 1-forms θ^p—linear combinations of these differentials:

$$\omega_i^p = c_{iq}^p \theta^q. \tag{5.57}$$

Under admissible linear transformations of the basis forms θ^p, the matrices $C_i = (c_{iq}^p)$ are transformed according to the tensor law with respect to the indices p and q.

A point $F \in L \subset Y$ is called a *focus* of a generator L if $dF \in L$ under some condition on the basis forms θ^p. In order to find the foci, we represent them in the form $F = x^i A_i$. Then

$$dF \equiv x^i \omega_i^p A_p \pmod{L},$$

and as a result, the foci are determined by the system of equations

$$x^i \omega_i^p = 0.$$

By (5.57), this system takes the form

$$x^i c_{iq}^p \theta^q = 0. \tag{5.58}$$

This system has a nontrivial solution with respect to the forms θ^q if and only if

$$\det(x^i c_{iq}^p) = 0. \tag{5.59}$$

Equation (5.59) determines on L the focus hypersurface F_L, which is an algebraic hypersurface of degree r.

Suppose that the point A_0 of our moving frame does not belong to the hypersurface F_L. Then the 1-forms ω_0^p are linearly independent, and we can take these forms as basis forms of the congruence Y. As a result, equations (5.57) become

$$\omega_a^p = c_{aq}^p \omega_0^q, \tag{5.60}$$

where $a = 1, \ldots, l$, and $c_{0q}^p = \delta_q^p$. Now equations (5.60) coincide with equations (3.11). As a result, equation (5.59) of the focus hypersurface F_L takes the form

$$\det(x^0 \delta_q^p + x^a c_{aq}^p) = 0. \tag{5.61}$$

Equation (5.61) coincides with equation (3.21) defining the foci on a plane generator L of a variety X with a degenerate Gauss map of rank r. However, unlike in Chapter 3, the quantities c_{aq}^p are not connected by any relations of type (3.9), because now there is no matrices $B^\alpha = (b_{pq}^\alpha)$. Thus, the focus hypersurfaces F_L determined by equation (5.61) are arbitrary determinant varieties (see Section 1.5.2, pp. 44–46) on generators L of the congruence Y in question.

In particular, if $l = 1$ and $n = r + 1$, then Y becomes a rectilinear congruence. Equation (5.61) defining the focus hypersurfaces F_L of such a congruence becomes

$$\det(x^0 \delta_q^p + x^1 c_{1q}^p) = 0. \tag{5.62}$$

Hence, each of the focus hypersurfaces F_L of Y decomposes into r real or complex points if each is counted as many times as its multiplicity. Each of these points describes a *focal variety* in $\mathbb{P}^n$ tangent to the generators L of the congruence Y. Recall that we encountered a similar situation in Section 5.1 (see p. 186) when we studied lightlike hypersurfaces in the de Sitter space.

Next, we consider a pseudocongruence Y^* in the space $\mathbb{P}^n$. Its generator L^* is of dimension $r-1$ and depends on r parameters. We place the points A_p,

$p = l+1, \ldots, n$, $l = n-r$, of our moving frame into the generator $L^* \subset Y^*$ and place the points A_i, $i = 0, 1, \ldots, l$, outside of L^*. The equations of infinitesimal displacement of such frames again have the form (5.56) but now the 1-forms ω_p^i are linear combinations of the basis forms θ^p defining a displacement of the generator $L^* = A_{l+1} \wedge \ldots \wedge A_n$. So now we have

$$\omega_p^i = b_{pq}^i \theta^q \tag{5.63}$$

and

$$dA_p = \omega_p^q A_q + b_{pq}^i \theta^q A_i. \tag{5.64}$$

Consider a hyperplane ξ passing through the generator $L^* \subset Y^*$. Relative to our moving frame, the equation of ξ is $\xi_i x^i = 0$, where ξ_i are tangential coordinates of the hyperplane ξ. The hyperplane ξ, which in addition to the generator L^* contains also a near generator $'L^*$ determined by the points A_p and dA_p, is called the *focus hyperplane.* By (5.64), the conditions defining the focus hyperplane are

$$\xi_i b_{pq}^i \theta^q = 0. \tag{5.65}$$

The system of equations (5.65) defines a displacement of the generator L^* if and only if this system has a nontrivial solution with respect to the forms θ^q. The necessary and sufficient condition for existence of such a nontrivial solution is the vanishing of the determinant of system (5.65):

$$\det(\xi_i b_{pq}^i) = 0. \tag{5.66}$$

Equation (5.66) defines the family of focus hyperplanes passing through the generator $L^* \subset Y^*$. This family is an algebraic hypercone of degree r whose vertex is the generator L^*. Note that equation (5.66) is similar to equation (3.24) of the focus hypercone Φ_L of a variety with a degenerate Gauss map.

5.2.2 Normalized Varieties in a Multidimensional Projective Space. Consider a smooth r-dimensional variety X in a projective space $\mathbb{P}^n$, $r < n$. The differential geometry on such a variety is rather poor. It is less rich than the differential geometry on varieties of the Euclidean space $\mathbb{E}^n$ or the spaces of constant curvature $\mathbb{S}^n$ and $\mathbb{H}^n$. With a first-order neighborhood of a point $x \in X \subset \mathbb{P}^n$, only the tangent subspace $T_x(X)$ is associated. As we saw in Section 1.4, where we studied a curve in the projective plane $\mathbb{P}^2$, in order to enrich the differential geometry, it is necessary to use differential prolongations of rather higher orders of the curve equations.

However, we can enrich the differential geometry of $X \subset \mathbb{P}^n$ if we endow X with an additional construction consisting of a subspace $N_x(X)$ of dimension $n - r$ such that $T_x(X) \cap N_x(X) = x$, and an $(r-1)$-dimensional subspace

$K_x(X)$, $K_x(X) \subset T_x(X)$, $x \notin K_x(X)$. We shall denote these subspaces simply by N_x and K_x and call the *normals of the first and second kind* (or simply the *first* and *second normals*) of the variety X, respectively (see the book by Norden [N 76], p. 198). The family of first normals forms a *congruence* N, and the family of second normals forms a *pseudocongruence* K in the space $\mathbb{P}^n$. If at any point of $x \in X$, there are assigned a single first normal N_x and a single second normal K_x, then the variety X is called *normalizated* (cf. Norden [N 76], p. 198, and Akivis and Goldberg [AG 93], Chapter 6).

As we will see below, on varieties of the Euclidean space $\mathbb{E}^n$ and the non-Euclidean spaces $\mathbb{S}^n$ and $\mathbb{H}^n$, the first and second normals are determined by the geometry of these spaces while on varieties of the affine space $\mathbb{A}^n$ and the projective space $\mathbb{P}^n$, these normals should be assigned artificially, or to find them, one should use higher order neighborhoods of a point $x \in X$. In this section, we shall apply the first method. Note that the second method is connected with great computational difficulties. One can find more details on this method and a related bibliography in the books [AG 93] by Akivis and Goldberg, Chapters 6, 7, and Norden [N 76], Chapter 5.

Thus, we consider now a normalized variety X of dimension r, $r = \dim X$, in the projective space $\mathbb{P}^n$. We associate with X a family of projective frames $\{A_0, A_1, \ldots, A_n\}$ in such a way that $A_0 = x, A_a \in N_x$, $a = 1, \ldots, l$, where $l = n - r$, and $A_p \in K_x$, $p = l + 1, \ldots, n$. The equations of infinitesimal displacement of these frames have the form

$$\begin{cases} dA_0 = \omega_0^0 A_0 & +\omega^p A_p, \\ dA_a = \omega_a^0 A_0 + \omega_a^b A_b & +\omega_a^p A_p, \\ dA_p = \omega_p^0 A_0 + \omega_p^a A_a & +\omega_p^q A_q, \end{cases} \tag{5.67}$$

Equations (5.67) show that for the family of moving frames in question, the system of differential equations

$$\omega^a = 0 \tag{5.68}$$

is satisfied, and the 1-forms ω^p are basis forms, because they determine a displacement of the point $A_0 = x$ along the variety X. Exterior differentiation of equations (5.68) and application of Cartan's lemma lead to the following equations:

$$\omega_p^a = b_{pq}^a \omega^q, \;\; b_{pq}^a = b_{qp}^a \tag{5.69}$$

(cf. equations (2.11) in Section 2.1). As we saw in Section 2.1, the quantities b_{pq}^a form a tensor and are coefficients of the second fundamental forms of the variety X at the point x:

$$\Phi^a = b_{pq}^a \omega^p \omega^q. \tag{5.70}$$

The points A_p belong to the tangent subspace $T_x(X)$. We assume that these points belong to the second normal $K_x \subset T_x(X)$, $K_x = A_{l+1} \wedge \ldots \wedge A_n$. Then, for $\omega^p = 0$, the 1-forms ω_p^0 must also vanish, and as a result, we have

$$\omega_p^0 = l_{pq}\omega^q. \tag{5.71}$$

Next, we place the points A_a of our moving frame into the first normal N_x of X, $N_x = A_0 \wedge A_1 \wedge \ldots \wedge A_l$. Then, for $\omega^p = 0$, we obtain that $\omega_a^p = 0$, and hence

$$\omega_a^p = c_{aq}^p\omega^q. \tag{5.72}$$

Consider a point $y \in N_x$ on the first normal. For this point, we have $y = y^0 A_0 + y^a A_a$. Differentiating this point by means of (5.67), we find that

$$dy = (dy^0 + y^0\omega_0^0 + y^a\omega_a^0)A_0 + (y^0\omega^p + y^a\omega_a^p)A_p + (dy^a + y^b\omega_b^a)A_a. \tag{5.73}$$

A point y is a *focus* of the first normal N_x if $dy \in N_x$. By (5.73), this condition implies that

$$y^0\omega^p + y^a\omega_a^p = 0.$$

Applying relations (5.72), we find that

$$(y^0\delta_q^p + y^a c_{aq}^p)\omega^q = 0.$$

This system has a nontrivial solution with respect to the forms ω^q if and only if

$$\det(y^0\delta_q^p + y^a c_{aq}^p) = 0. \tag{5.74}$$

Equation (5.74) differs from equation (5.61) only in notation, and it defines the focus hypersurface F_x in the generator N_x of the congruence of first normals associated with the variety X. It follows from equation (5.74) that the point $x \in X$, whose coordinates are $y^0 = 1$, $y^a = 0$, does not belong to the focus hypersurface F_x.

Let us find the focus hypercones Φ_x of the pseudocongruence K of second normals of X. The hypercones Φ_x are formed by the hyperplanes ξ of the space $\mathbb{P}^n$ containing the second normal $K_x = A_{l+1} \wedge \ldots \wedge A_n \subset T_x(X)$ and its neighboring normal $K_x + dK_x$, which contains not only the points A_p but also the points

$$dA_p \equiv \omega_p^0 A_0 + \omega_p^a A_a \pmod{N_x}.$$

As a result, tangential coordinates ξ_0 and ξ_a of such a hyperplane satisfy the equations

$$\xi_0\omega_p^0 + \xi_a\omega_p^a = 0.$$

By (5.71) and (5.72), it follows from this equation that

$$(\xi_0 l_{pq} + \xi_a b^a_{pq})\, \omega^q = 0.$$

This system has a nontrivial solution with respect to the forms ω^q if and only if its determinant vanishes,

$$\det(\xi_0 l_{pq} + \xi_a b^a_{pq}) = 0. \tag{5.75}$$

Equation (5.75) determines an algebraic hypercone of order r whose vertex is the generator K_x of the pseudocongruence K of the second normals. This hypercone is called the *focus hypercone* of the pseudocongruence K.

Next, we consider the tangent and normal bundles associated with a normalized variety X. The base of both bundles is the variety X itself, the fibers of the tangent bundle are the tangent subspaces T_x, and the fibers of the normal bundle are the second normals N_x.

Suppose that $'x = x + x^p A_p$ is an arbitrary point in the tangent subspace T_x, and $\boldsymbol{x} = {}'x - x = x^p A_p$ is a vector in the tangent bundle TX. The differential of this vector has the form

$$d\boldsymbol{x} = (dx^p + x^q \omega^p_q) A_p + x^p (l_{pq} A_0 + b^a_{pq} A_a) \omega^q. \tag{5.76}$$

The first term on the right-hand side of (5.76) belongs to the tangent subspace T_x, and the second term belongs to N_x. The 1-form $Dx^p = dx^p + x^q \omega^p_q$ is called the *covariant differential* of the vector field $\boldsymbol{x} = (x^p)$. The vector field $\boldsymbol{x}$ is called *parallel* on the tangent bundle $T(X)$ if the form Dx^p vanishes, i.e., if

$$Dx^p = dx^p + x^q \omega^p_q = 0. \tag{5.77}$$

The 1-forms ω^p_q are the components of the *connection form* $\omega = \{\omega^p_q\}$ of the affine connection on the variety X.

We find the exterior differentials of the components ω^p_q of the connection form ω. By (5.69), (5.71), and (5.72), these exterior differentials have the form

$$d\omega^p_q = \omega^s_q \wedge \omega^p_s + (l_{qs} \delta^p_t + b^a_{qs} c^p_{at}) \omega^s \wedge \omega^t. \tag{5.78}$$

The 2-form

$$\Omega^p_q = d\omega^p_q - \omega^s_q \wedge \omega^p_s$$

is said to be *the curvature form* of the affine connection on the variety X. From equation (5.78) it follows that

$$\Omega^p_q = \frac{1}{2} R^p_{qst} \omega^s \wedge \omega^t, \tag{5.79}$$

where

$$R^p_{qst} = l_{qs}\delta^p_t + b^a_{qs}c^p_{at} - l_{qt}\delta^p_s - b^a_{qt}c^p_{as} \tag{5.80}$$

(cf. formula (6.25) on p. 179 of the book [AG 93] by Akivis and Goldberg) is the *curvature tensor* of the affine connection on X. Equations (5.80) allow us to compute the curvature tensor for different normalizations of the variety X.

If $R^p_{qst} = 0$ on the variety X, then the affine connection on X is *flat*, and a parallel translation of a vector $\boldsymbol{x}$ does not depend on the path of integration (see, for example, Norden [N 76], p. 118, or Kobayashi and Nomizu [KN 76], p. 70).

Further, we consider a vector field $\boldsymbol{y}$ in the normal bundle $N(X)$. This vector is determined by the point x and a point $y = y^0 A_0 + y^a A_a$ of the fiber $N_x \subset N(X)$. The differential of the point y is defined by equation (5.73).

The 1-form

$$Dy^a = dy^a + y^b\omega^a_b \tag{5.81}$$

is called the *covariant differential* of the vector field $\boldsymbol{y}$ in the normal bundle $N(X)$, and the forms ω^b_a are the components of the *connection form of the normal connection* on a normalized variety X (see, for example, Cartan [C 01], p. 242; see more on the normal connection in the paper [AG 95] and Section 6.3 of the book [AG 93] by Akivis and Goldberg). The 2-form

$$\Omega^a_b = d\omega^a_b - \omega^c_b \wedge \omega^a_c$$

is called the *curvature form* of the normal connection. Note that Cartan in [C 01] called this form the *Gaussian torsion* of an embedded variety X.

Differentiating the forms ω^a_b and applying formulas (5.69) and (5.72), we find the expression of the curvature form Ω^a_b:

$$\Omega^a_b = \frac{1}{2}R^a_{bst}\omega^s \wedge \omega^t, \tag{5.82}$$

where

$$R^a_{bst} = c^p_{bs}b^a_{pt} - c^p_{bt}b^a_{ps}. \tag{5.83}$$

The tensor R^a_{bst} is called the *tensor of normal curvature* of the variety X.

The second normals K_x associated with the variety X allow us to find a distribution Δ_y of r-dimensional subspaces associated with X. The elements of the distribution Δ_y are linear spans of the points $y \in N_x$ and the second normals K_x, $\Delta_y = y \wedge K_x$. By (5.73), the distribution Δ_y is determined by the system of equations

$$dy^a + y^b\omega^a_b = 0. \tag{5.84}$$

In the general case, the system of equations (5.84) is not completely integrable, and when a point x moves along a closed contour $l \subset X$, the corresponding point y does not describe a closed contour.

But the point y describes a closed contour l' if system (5.84) is completely integrable. The condition of complete integrability of (5.84) is the vanishing of the tensor of normal curvature (5.83) of the variety X. In this case, the distribution Δ_y defined by system (5.84) is completely integrable, and the closed contours l' lie on integral varieties of this distribution. These integral varieties form an $(n-r)$-parameter family of r-dimensional subvarieties $X(y)$ which are "parallel" to the variety X in the sense that the subspaces $T_x(X)$ and $T_x(X(y))$ pass through the same second normal K_x.

Now suppose that a normalized variety $X \subset \mathbb{P}^n$ has a flat normal connection, i.e., $R^a_{bst} = 0$. By (5.83), these conditions lead to the relation

$$b^a_{pt} c^p_{bs} = b^a_{ps} c^p_{bt}. \tag{5.85}$$

Relations (5.85) differ from relations (3.9) (p. 94) in Chapter 3 only in notation. If we introduce the matrix notations

$$B^a = (b^a_{pq}), \;\; C_b = (c^p_{bq})$$

(cf. Section 3.1, p. 94), then relations (5.85) take the form

$$(B^a C_b) = (B^a C_b)^T \tag{5.86}$$

(cf. (3.12), p. 94).

We proved in Chapters 3 and 4 that these relations imply that the matrices B^a and C_b can be simultaneously reduced to a diagonal form or a block diagonal form. Therefore, *the focus hypersurfaces $F_x \subset N_x$ of the variety X decompose into the plane generators of different dimensions* (see Chapters 3 and 4). This property of the varieties X with a flat normal connection allows us to construct a classification of such varieties in the same way as this was done for the varieties with degenerate Gauss maps in a projective space. For varieties in an affine space and a Euclidean space, such a classification was outlined in the papers [ACh 75, 76, 01] by Akivis and Chakmazyan.

5.2.3 Normalization of Varieties of Affine and Euclidean Spaces. An affine space $\mathbb{A}^n$ differs from a projective space $\mathbb{P}^n$ by the fact that in $\mathbb{A}^n$ a hyperplane at infinity $\mathbb{P}_\infty$ is fixed. If we place the points A_i, $i = 1, \ldots, n$, of our moving projective frame into this hyperplane, then the equations of infinitesimal displacement of the moving frame take the form (1.81),

$$\begin{cases} dA_0 = \omega^0_0 A_0 + \omega^i_0 A_i, \\ dA_i = \qquad\qquad \omega^j_i A_j, \;\; i, j = 1, \ldots, n \end{cases} \tag{5.87}$$

(see p. 25), and the structure equations of the affine space $\mathbb{A}^n$ take the form

$$d\omega_0^0 = 0, \quad d\omega_0^i = \omega_0^j \wedge \omega_j^i, \quad d\omega_j^i = \omega_j^k \wedge \omega_k^i. \tag{5.88}$$

Consider a variety X of dimension r in the affine space $\mathbb{A}^n$. The tangent space $T_x(X)$ intersects the hyperplane at infinity $\mathbb{P}_\infty$ in a subspace K_x of dimension $r-1$, $K_x = T_x \cap \mathbb{P}_\infty$. Thus, for a normalization of X, it is sufficient to assign only a family of first normals N_x. If we place the points A_a, $a = 1, \ldots, l$, of our moving frame into the subspace $N_x \cap \mathbb{P}_\infty$, and the points A_p, $p = l+1, \ldots, n$, into the subspace K_x, then equations (5.87) take the form

$$\begin{cases} dA_0 = \omega_0^0 A_0 & +\omega_0^p A_p, \\ dA_a = \qquad \omega_a^b A_b & +\omega_a^p A_p, \\ dA_p = \qquad \omega_p^a A_a & +\omega_p^q A_q \end{cases} \tag{5.89}$$

(cf. equations (5.67)).

As was in the projective space, we have the equations (5.69),

$$\omega_p^a = b_{pq}^a \omega^q, \quad b_{pq}^a = b_{qp}^a, \tag{5.90}$$

where b_{pq}^a is the second fundamental tensor of the variety X. Equations (5.72) also preserve their form:

$$\omega_a^p = c_{aq}^p \omega^q, \tag{5.91}$$

but equations (5.71) become

$$\omega_p^0 = 0. \tag{5.92}$$

Thus $l_{pq} = 0$, and the equation of the focus hypersurface $F_x \subset N_x$ preserves its form (5.74):

$$\det(y^0 \delta_q^p + y^a c_{aq}^p) = 0. \tag{5.93}$$

As to equation (5.75) of the focus hypercone Φ_x, by (5.92), this equation takes the form

$$\det(\xi_a b_{pq}^a) = 0. \tag{5.94}$$

Expressions (5.80) for the components of the curvature tensor of the affine connection induced on the normalized variety $X \subset \mathbb{A}^n$ take now the form

$$R_{qst}^p = b_{qs}^a c_{at}^p - b_{qt}^a c_{as}^p, \tag{5.95}$$

and the expression (5.83) for the components of the tensor of normal curvature of the variety X preserves its form:

$$R_{bst}^a = b_{pt}^a c_{bs}^p - b_{ps}^a c_{bt}^p. \tag{5.96}$$

As was the case in the projective space, the vanishing of tensor of normal curvature R^a_{bst} is equivalent to the complete integrability of the system defining the distribution $\Delta_y = y \wedge K_x$, where $y \in N_x$. But in the affine space, the elements Δ_y of this distribution are parallel to the subspace $T_x(X)$. As a result, *a variety $X \subset \mathbb{A}^n$ has a flat normal connection if and only if this variety admits an l-parameter family of parallel varieties $X(y)$, where $y \in N_x$.*

Further consider a variety X of dimension r in the Euclidean space $\mathbb{E}^n$. On X, both the second normal $K_x = T_x \cap \mathbb{P}_\infty$ and the first normal N_x orthogonal to the tangent subspace $T_x(X)$ are naturally defined.

In the Euclidean space $\mathbb{E}^n$, there is defined a scalar product of vectors, and a scalar product of points in the hyperplane at infinity $\mathbb{P}_\infty$ is induced by the scalar product in $\mathbb{E}^n$. Because in our moving frame, we have $A_a \in N_x \cap \mathbb{P}_\infty$; $A_p \in T_x \cap \mathbb{P}_\infty = K_x$, $a = 1, \ldots, l$; $p = l+1, \ldots, n$; and $T_x \perp N_x$, we find that

$$(A_a, A_p) = 0, \tag{5.97}$$

where, as usually, the parentheses denote the scalar product of points in the hyperplane at infinity $\mathbb{P}_\infty$. In addition, we set

$$(A_a, A_b) = g_{ab}, \;\; (A_p, A_q) = g_{pq}, \tag{5.98}$$

where g_{ab} and g_{pq} are nondegenerate symmetric tensors.

Differentiating equations (5.97) and using formulas (5.89), (5.97) and (5.98), we find that

$$g_{ab}\, \omega^b_p + g_{pq}\, \omega^q_a = 0.$$

It follows that

$$\omega^p_a = -g^{pq}\, g_{ab}\, \omega^b_q. \tag{5.99}$$

Equations (5.99) and (5.90) imply that

$$\omega^p_a = -g^{pq}\, g_{ac}\, b^c_{qs} \omega^s. \tag{5.100}$$

Comparing (5.100) and (5.91), we obtain

$$c^p_{as} = -g^{pq}\, g_{ac}\, b^c_{qs}. \tag{5.101}$$

Now we find the equation of the focus hypersurface F_x of the variety $X \in \mathbb{E}^n$. By (5.93) and (5.101), we have the following equation for F_x:

$$\det(y^0 \delta^p_q - y^a g^{ps} g_{ac} b^c_{sq}) = 0.$$

The last equation is equivalent to the equation

$$\det(y^0 g_{pq} - y_a b^a_{pq}) = 0, \tag{5.102}$$

where $y_a = g_{ab}y^b$.

In our moving frame, the hyperplane at infinity $\mathbb{P}_\infty$ is determined by the equation $y^0 = 0$. Hence by (5.102), the intersection $F_x \cap \mathbb{P}_\infty$ of the focus hypersurface F_x with the hyperplane at infinity $\mathbb{P}_\infty$ is defined by the equation

$$\det(y_a b^a_{pq}) = 0. \tag{5.103}$$

But this equation differs only in notation from equation (5.94) of the focus hypercone Φ_x of the variety X. Equations (5.94) and (5.103) coincide if $\xi_a = y_a = g_{ab}y^b$. This means that *the focus hypercone* Φ_x *of the variety* $X \subset \mathbb{E}^n$ *is formed by the hyperplanes* ξ *containing the tangent subspace* T_x *and orthogonal at the points* $\widetilde{y}$ *of the hyperplane at infinity* $\mathbb{P}_\infty$ *lying in the intersection* $F_x \cap \mathbb{P}_\infty$.

This result clarifies the geometric meaning of the focus hypercone Φ_x for the variety $X \subset \mathbb{E}^n$ and its relation with the focus hypersurface F_x of X.

We also find the curvature tensor of the affine connection induced on the variety $X \subset \mathbb{E}^n$. Substituting the values of c^p_{aq} from (5.101) into formula (5.95), we find that

$$R^p_{qst} = g^{pu} g_{ac}(b^a_{qt} b^c_{us} - b^a_{qs} b^c_{ut}). \tag{5.104}$$

Contracting equation (5.104) with the tensor g_{pv} and changing the summation indices (if necessary), we find that

$$R_{pqst} = g_{ac}(b^a_{ps} b^c_{qt} - b^a_{pt} b^c_{qs}), \tag{5.105}$$

where $R_{pqst} = g_{pu} R^u_{qst}$. Formulas (5.104) and (5.105) give the usual expressions for the curvature tensor of the affine connection induced on a normalized variety $X \subset \mathbb{E}^n$.

But in addition to the curvature tensor of the affine connection induced on a normalized variety $X \subset \mathbb{E}^n$, we considered also the tensor R^a_{bst} of normal curvature defined by equation (5.96). Substituting the values of c^p_{aq} from (5.101) into formula (5.96), we find that

$$R^a_{bst} = g^{pq} g_{bc}(b^c_{qt} b^a_{ps} - b^c_{qs} b^a_{pt}). \tag{5.106}$$

As we noted earlier, in the book [C 01] by É. Cartan, the exterior 2-form

$$\Omega^a_b = d\omega^a_b - \omega^c_b \wedge \omega^a_c = \frac{1}{2} R^a_{bst} \omega^s \wedge \omega^t$$

is called the *Gaussian torsion* of a variety $X \subset \mathbb{E}^n$.

5.3 Varieties with Degenerate Gauss Maps Associated with Smooth Lines on Projective Planes over Two-Dimensional Algebras

5.3.1 Two-Dimensional Algebras and Their Representations. There are three known two-dimensional algebras: the algebra of complex numbers $z = x + iy$, where $i^2 = -1$; the algebra of double (or split complex) numbers $z = x + ey$, where $e^2 = 1$; and the algebra of dual numbers $z = x + \varepsilon y$, where $\varepsilon^2 = 0$. Here everywhere $x, y \in \mathbb{R}$. Usually these three algebras are denoted by $\mathbb{C}, \mathbb{C}^1$, and $\mathbb{C}^0$, respectively (see Rosenfeld [Ro 97], §1.1). These algebras are commutative and associative, and any two-dimensional algebra is isomorphic to one of them.

Each of these three algebras admits a representation by means of the real (2×2)-matrices:

$$z = x + iy \to \begin{pmatrix} x & -y \\ y & x \end{pmatrix}, \tag{5.107}$$

$$z = x + ey \to \begin{pmatrix} x & y \\ y & x \end{pmatrix}, \tag{5.108}$$

and

$$z = x + \varepsilon y \to \begin{pmatrix} x & 0 \\ y & x \end{pmatrix}. \tag{5.109}$$

In what follows, we will identify the algebras $\mathbb{C}, \mathbb{C}^1$, and $\mathbb{C}^0$ with their matrix representations.

The algebras $\mathbb{C}, \mathbb{C}^1$, and $\mathbb{C}^0$ are subalgebras of the complete matrix algebra $\mathbb{M}$ formed by all real (2×2)-matrices

$$\begin{pmatrix} x_0^0 & x_1^0 \\ x_0^1 & x_1^1 \end{pmatrix}, \tag{5.110}$$

which is associative but not commutative.

The algebra $\mathbb{C}$ does not have zero divisors while the algebras $\mathbb{C}^1, \mathbb{C}^0$, and $\mathbb{M}$ have such divisors. In the matrix representation, zero divisors of these algebras are determined by the condition

$$\det \begin{pmatrix} x_0^0 & x_1^0 \\ x_0^1 & x_1^1 \end{pmatrix} = 0.$$

For the algebra $\mathbb{C}^1$ the last condition takes the form

$$x^2 - y^2 = 0,$$

for the algebra $\mathbb{C}^0$ the form $x = 0$, and for the algebra $\mathbb{M}$ the form

$$x_0^0 x_1^1 - x_0^1 x_1^0 = 0. \tag{5.111}$$

The elements of the algebras $\mathbb{C}^1$ and $\mathbb{C}^0$, as well as the regular complex numbers (the elements of the algebra $\mathbb{C}$), can be represented by the points on the plane xOy. In this representation, the zero divisors of the algebra $\mathbb{C}^1$ are represented by the points of the straight lines $y = \pm x$, and the zero divisors of the algebra $\mathbb{C}^0$ by the points of the y-axis.

The elements of the algebra $\mathbb{M}$ are represented by the points of a four-dimensional vector space, and its zero divisors by the points of the cone (5.111) whose signature is $(2, 2)$. Thus, to the algebra $\mathbb{M}$, there corresponds a four-dimensional pseudo-Euclidean space E_2^4 of signature 2 with the isotropic cone (5.111). This cone bears two families of plane generators defined by the equations

$$\frac{x_0^0}{x_0^1} = \frac{x_1^0}{x_1^1} = \lambda, \qquad \frac{x_0^0}{x_1^0} = \frac{x_0^1}{x_1^1} = \mu, \tag{5.112}$$

where λ and μ are real numbers.

5.3.2 The Projective Planes over the Algebras $\mathbb{C}, \mathbb{C}^1$, $\mathbb{C}^0$, and $\mathbb{M}$. Denote by $\mathbb{A}$ one of the algebras $\mathbb{C}, \mathbb{C}^1, \mathbb{C}^0$, or $\mathbb{M}$ and consider a projective plane $\mathbb{A}\mathbb{P}^2$ over the algebra $\mathbb{A}$ (see Bourbaki [Bou 70]). A point $Y \in \mathbb{A}\mathbb{P}^2$ has three matrix coordinates Y^0, Y^1, Y^2 that have, respectively, the form (5.107), (5.108), (5.109), or (5.110). Because it is convenient to write point coordinates as a column-matrix, we write

$$Y = (Y^0, Y^1, Y^2)^T. \tag{5.113}$$

The matrix Y in (5.113) has six rows and two columns. Of course, the columns of this matrix must be linearly independent. The coordinates Y^α, $\alpha = 0, 1, 2$, are defined up to a multiplication from the right by an element P of the algebra $\mathbb{A}$, which is not a zero divisor. So we have $Y' \sim YP$, $P \in \mathbb{A}$.

In particular, for $Y \in \mathbb{C}\mathbb{P}^2$, $Y \in \mathbb{C}^1\mathbb{P}^2$, and $Y \in \mathbb{C}^0\mathbb{P}^2$, we have

$$Y = \begin{pmatrix} y_0^0 & -y_0^1 \\ y_0^1 & y_0^0 \\ y_0^2 & -y_0^3 \\ y_0^3 & y_0^2 \\ y_0^4 & -y_0^5 \\ y_0^5 & y_0^4 \end{pmatrix}, \quad Y = \begin{pmatrix} y_0^0 & y_0^1 \\ y_0^1 & y_0^0 \\ y_0^2 & y_0^3 \\ y_0^3 & y_0^2 \\ y_0^4 & y_0^5 \\ y_0^5 & y_0^4 \end{pmatrix}, \quad Y = \begin{pmatrix} y_0^0 & 0 \\ y_0^1 & y_0^0 \\ y_0^2 & 0 \\ y_0^3 & y_0^2 \\ y_0^4 & 0 \\ y_0^5 & y_0^4 \end{pmatrix},$$

respectively.

The columns of the matrix Y can be considered as coordinates of the points y_0 and y_1 of a real projective space $\mathbb{RP}^5$, and to the matrix Y there corresponds the straight line $y_0 \wedge y_1$ in the space $\mathbb{RP}^5$. So we can set $Y = y_0 \wedge y_1$. The set of all straight lines of the space $\mathbb{RP}^5$ forms the Grassmannian $\mathbb{RG}(1,5)$, whose dimension is equal to eight, $\dim \mathbb{RG}(1,5) = 2 \cdot 4 = 8$.

Note that $\mathbb{RG}(1,5)$ is a differentiable manifold. Thus, $\mathbb{AP}^2$ is also a differentiable manifold over $\mathbb{R}$.

5.3.3 Equation of a Straight Line. A straight line U in the plane $\mathbb{AP}^2$ is defined by the equation

$$U_0 Y^0 + U_1 Y^1 + U_2 Y^2 = 0,$$

where $U_\alpha \in \mathbb{A}$, $\alpha = 0, 1, 2$. The coordinates U_α admit a multiplication from the left by an element $P \in \mathbb{A}$, which is not a zero divisor.

In general, two *skewed* straight lines in $\mathbb{RP}^5$ correspond to two points $Y, Z \in \mathbb{AP}^2$. These straight lines define a subspace $\mathbb{RP}^3$ corresponding to the unique straight line in $\mathbb{AP}^2$ passing through the points Y and Z.

Two points Y and Z are called *adjacent* if more than one straight line passes through them in $\mathbb{AP}^2$. To such points, there correspond *intersecting* straight lines $y^0 \wedge y^1$ and $z^0 \wedge z^1$ in $\mathbb{RP}^5$. Through adjacent points $Y, Z \in \mathbb{AP}^2$, there passes a two-parameter family of straight lines in $\mathbb{AP}^2$, because through a plane $\mathbb{RP}^5$, there passes a two-parameter family of subspaces $\mathbb{RP}^3 \subset \mathbb{RP}^5$.

If

$$Y = (Y^0, Y^1, Y^2)^T, \quad Z = (Z^0, Z^1, Z^2)^T$$

are adjacent points, then the rank of the (6×4)-matrix composed of the matrix coordinates of Y and Z is less than four. If the rank of this matrix is four, then through the points Y and Z, there passes a unique straight line.

On a plane $\mathbb{AP}^2$ there are three basis points E_0, E_1, E_2 with coordinates

$$E_0 = (E, 0, 0)^T, \quad E_1 = (0, E, 0)^T, \quad E_2 = (0, 0, E)^T,$$

where $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the unit matrix, and 0 is the 2×2 zero-matrix. A point $Y \in \mathbb{AP}^2$ can be represented in the form

$$Y = E_0 Y^0 + E_1 Y^1 + E_2 Y^2. \tag{5.114}$$

However, as we noted earlier, the coordinates Y_α of this point admit a multiplication from the right by an element $P \in \mathbb{A}$, which is not a zero divisor.

A point Y is in general position with the straight line $E_\alpha \wedge E_\beta$, $\alpha, \beta = 0, 1, 2$, if and only if its coordinate Y^γ, $\gamma \neq \alpha, \beta$, is not a zero divisor. Let, for example, a point Y be in general position with the straight line $E_1 \wedge E_2$. Then its coordinate Y^0 is not a zero divisor, and all its coordinates can be multiplied from the right by $(Y^0)^{-1}$. Then expression (5.114) of the point Y takes the form

$$Y = E_0 + E_1\widetilde{Y}^1 + E_2\widetilde{Y}^2, \tag{5.115}$$

where $\widetilde{Y}^1 = Y^1(Y^0)^{-1}, \widetilde{Y}^2 = Y^2(Y^0)^{-1}$. Now the (4×2)-matrix $(\widetilde{Y}^1, \widetilde{Y}^2)^T$ is defined uniquely and is called the *matrix coordinate* of the point Y as well as of the straight line $y_0 \wedge y_1$ defined in the space $\mathbb{RP}^5$ by the point Y (see Rosenfeld [Ro 97], Section 2.4.1, and also Rosenfeld [Ro 66], Chapter 3, §3).

For the plane $\mathbb{MP}^2$, the matrix coordinate has eight real components. Hence $\dim \mathbb{MP}^2 = 8$. Because $\dim \mathbb{MP}^2 = \dim \mathbb{RG}(1,5)$, the plane $\mathbb{MP}^2$ can be bijectively mapped onto the Grassmannian $\mathbb{RG}(1,5)$.

For the planes $\mathbb{CP}^2, \mathbb{C}^1\mathbb{P}^2$, and $\mathbb{C}^0\mathbb{P}^2$, the matrix coordinates of points have four real components. Hence the real dimension of these planes is four,

$$\dim \mathbb{CP}^2 = \dim \mathbb{C}^1\mathbb{P}^2 = \dim \mathbb{C}^0\mathbb{P}^2 = 4.$$

Therefore, the family of straight lines $y_0 \wedge y_1$ in the space $\mathbb{RP}^5$ for each of these planes depends on four parameters, i.e., it forms a *congruence* in the space $\mathbb{RP}^5$. We denote these congruences by K, K^1, and K^0, respectively.

5.3.4 Moving Frames in Projective Planes over Algebras. A moving frame in a projective plane over an algebra $\mathbb{A}$ is a triple of points A_α, $\alpha = 0, 1, 2$, that are mutually not adjacent. Any point $Y \in \mathbb{AP}^2$ can be written as

$$Y = A_0Y^0 + A_1Y^1 + A_2Y^2,$$

where $Y^\alpha \in \mathbb{A}$ are the coordinates of this point with respect to the frame $\{A_0, A_1, A_2\}$. The coordinates of a point Y are defined up to a multiplication from the right by an element P of the algebra $\mathbb{A}$ that is not a zero divisor. If a point Y is in general position with the straight line $A_1 \wedge A_2$, then its coordinate Y^0 is not a zero divisor. Thus, the point Y can be written as

$$Y = A_0 + A_1\widetilde{Y}^1 + A_2\widetilde{Y}^2,$$

where $\widetilde{Y}^1 = Y^1(Y^0)^{-1}, \widetilde{Y}^2 = Y^2(Y^0)^{-1}$. The matrix $(\widetilde{Y}^1, \widetilde{Y}^2)^T$ is the matrix coordinate of the point Y with respect to the moving frame $\{A_\alpha\}$, and this matrix coordinate is defined uniquely.

The plane $\mathbb{AP}^2$ admits a representation on the Grassmannian $\mathbb{RG}(1,5)$ formed by the straight lines of the space $\mathbb{RP}^5$. Under this representation, the straight lines

$$A_0 = a_0 \wedge a_1, \quad A_1 = a_2 \wedge a_3, \quad A_2 = a_4 \wedge a_5 \tag{5.116}$$

in $\mathbb{RP}^5$ correspond to the vertices of the frame $\{A_\alpha\}$; here a_i, $i = 0, \dots, 5$, are points of the space $\mathbb{RP}^5$.

The equations of infinitesimal displacement of the moving frame $\{A_0, A_1, A_2\}$ have the form

$$dA_\alpha = A_\beta \Omega_\alpha^\beta, \quad \alpha, \beta = 0, 1, 2, \tag{5.117}$$

where Ω_α^β are 1-forms over the algebra $\mathbb{A}$. In the representation of the algebra $\mathbb{A}$ by (2×2)-matrices, these forms are expressed as the transposed matrices (5.107), (5.108), (5.109), and (5.110). Their entries are not the numbers. They are real 1-forms:

$$\Omega_\alpha^\beta = \begin{pmatrix} \omega_{2\alpha}^{2\beta} & \omega_{2\alpha}^{2\beta+1} \\ \omega_{2\alpha+1}^{2\beta} & \omega_{2\alpha+1}^{2\beta+1} \end{pmatrix}. \tag{5.118}$$

Thus, for the plane $\mathbb{CP}^2$, the entries of the matrix Ω_α^β satisfy the equations

$$\omega_{2\alpha}^{2\beta} = \omega_{2\alpha+1}^{2\beta+1}, \quad \omega_{2\alpha}^{2\beta+1} = -\omega_{2\alpha+1}^{2\beta}, \tag{5.119}$$

for the plane $\mathbb{C}^1\mathbb{P}^2$ the equations

$$\omega_{2\alpha}^{2\beta} = \omega_{2\alpha+1}^{2\beta+1}, \quad \omega_{2\alpha}^{2\beta+1} = \omega_{2\alpha+1}^{2\beta}, \tag{5.120}$$

and for the plane $\mathbb{C}^0\mathbb{P}^2$ the equations

$$\omega_{2\alpha}^{2\beta} = \omega_{2\alpha+1}^{2\beta+1}, \quad \omega_{2\alpha+1}^{2\beta} = 0. \tag{5.121}$$

If the frame $\{A_\alpha\}$ moves in the plane $\mathbb{AP}^2$, then the points $a_i \in \mathbb{RP}^5$ also move. The equations of infinitesimal displacement of the moving frame $\{a_i\}$ can be written in the form

$$da_i = a_j \omega_i^j, \quad i, j = 0, 1, \dots, 5, \tag{5.122}$$

where by (5.116) the forms ω_i^j coincide with the corresponding forms (5.118). The forms ω_j^i satisfy the structure equations of the projective space $\mathbb{RP}^5$:

$$d\omega_j^i = -\omega_k^i \wedge \omega_j^k, \tag{5.123}$$

where d is the symbol of exterior differential, and $\wedge$ denotes the exterior multiplication of the linear differential forms (see Section 1.2.4).

5.3.5 Focal Properties of the Congruences K, K^1, and K^0. Now we consider the congruences K, K^1, and K^0 of the space $\mathbb{RP}^5$, representing the planes $\mathbb{CP}^2, \mathbb{C}^1\mathbb{P}^2$, and $\mathbb{C}^0\mathbb{P}^2$ in this space, and investigate their focal properties.

Theorem 5.8. *The projective planes $\mathbb{CP}^2, \mathbb{C}^1\mathbb{P}^2$, and $\mathbb{C}^0\mathbb{P}^2$ admit a bijective mapping onto the linear congruences K, K^1, and K^0 of the real space $\mathbb{RP}^5$. These congruences are, respectively, elliptic, hyperbolic, and parabolic.*

Proof. To each of these congruences, we associate a family of projective frames in such a way that the points a_0 and a_1 are located on a moving straight line of the congruence.

For the congruence K, equations of infinitesimal displacement of the points a_0 and a_1 can be written in the form

$$\begin{cases} da_0 = \omega_0^0 a_0 + \omega_0^1 a_1 + \omega_0^2 a_2 + \omega_0^3 a_3 + \omega_0^4 a_4 + \omega_0^5 a_5, \\ da_1 = -\omega_1^0 a_0 + \omega_0^0 a_1 - \omega_0^3 a_2 + \omega_0^2 a_3 - \omega_0^5 a_4 + \omega_0^4 a_5. \end{cases} \tag{5.124}$$

By (5.120), for the congruence K^1, these two equations take the form

$$\begin{cases} da_0 = \omega_0^0 a_0 + \omega_0^1 a_1 + \omega_0^2 a_2 + \omega_0^3 a_3 + \omega_0^4 a_4 + \omega_0^5 a_5, \\ da_1 = \omega_0^1 a_0 + \omega_0^0 a_1 + \omega_0^3 a_2 + \omega_0^2 a_3 + \omega_0^5 a_4 + \omega_0^4 a_5. \end{cases} \tag{5.125}$$

Finally, by (5.121), for the congruence K^0, these two equations take the form

$$\begin{cases} da_0 = \omega_0^0 a_0 + \omega_0^1 a_1 + \omega_0^2 a_2 + \omega_0^3 a_3 + \omega_0^4 a_4 + \omega_0^5 a_5, \\ da_1 = \omega_0^0 a_1 + \omega_0^2 a_3 + \omega_0^4 a_5. \end{cases} \tag{5.126}$$

Let $x = a_1 + \lambda a_0$ be an arbitrary point of the straight line $a_0 \wedge a_1$. This point is a focus of this straight line if for some displacement, its differential dx also belongs to this straight line.

Let us start from the congruence K^1, because the focal images for this congruence are real and look more visual. By (5.125), for this congruence we have

$$\begin{aligned} dx \equiv\ & (\omega_0^3 + \lambda\omega_0^2) a_2 + (\omega_0^2 + \lambda\omega_0^3) a_3 + (\omega_0^5 + \lambda\omega_0^4) a_4 \\ & + (\omega_0^4 + \lambda\omega_0^5) a_5 \pmod{a_0 \wedge a_1}; \end{aligned} \tag{5.127}$$

as a result, for its focus x, the following equations must be satisfied:

$$\begin{cases} \omega_0^2 + \lambda\omega_0^3 = 0, \quad \omega_0^4 + \lambda\omega_0^5 = 0, \\ \lambda\omega_0^2 + \omega_0^3 = 0, \quad \lambda\omega_0^4 + \omega_0^5 = 0. \end{cases} \tag{5.128}$$

The necessary and sufficient condition of consistency of this system is

$$\begin{vmatrix} 1 & \lambda \\ \lambda & 1 \end{vmatrix}^2 = 0.$$

It follows that the values $\lambda = \pm 1$ are double roots of this equation. Thus, each line $a_0 \wedge a_1$ of the congruence K^1 has two double foci

$$f_1 = a_1 + a_0, \quad f_2 = a_1 - a_0.$$

Equations (5.127) imply that the differentials of the focus f_1 are expressed only in terms of the points $a_0 + a_1, a_2 + a_3$, and $a_4 + a_5$. The differentials of these points are expressed in terms of the same points. As a result, the plane

$$\pi_1 = (a_0 + a_1) \wedge (a_2 + a_3) \wedge (a_4 + a_5)$$

remains fixed when the straight line $a_0 \wedge a_1$ describes the congruence K^1 in the space $\mathbb{RP}^5$. In a similar way, one can prove that the focus f_2 describes the plane

$$\pi_2 = (a_0 - a_1) \wedge (a_2 - a_3) \wedge (a_4 - a_5).$$

Thus, the congruence K^1 is a four-parameter family of straight lines of the space $\mathbb{RP}^5$ intersecting its two planes π_1 and π_2 that are in general position. Hence K^1 is a *hyperbolic* line congruence.

In a similar way, we can prove that each straight line $a_0 \wedge a_1$ of the congruence K bears two double complex conjugate foci,

$$f_1 = a_1 + ia_0, \quad f_2 = a_1 - ia_0,$$

and these foci describe two complex conjugate two-dimensional planes π_1 and $\pi_2, \pi_2 = \overline{\pi}_1$. Hence K is an *elliptic* line congruence in the space $\mathbb{RP}^5$. The straight lines of K do not have real singular points in $\mathbb{RP}^5$.

Finally, consider the congruence K^0 in the space $\mathbb{RP}^5$. We look for the foci of its straight lines in the same form

$$x = a_1 + \lambda a_0.$$

Differentiating this expression by means of (5.126), we find that

$$dx \equiv \lambda\omega_0^2 a_2 + (\lambda\omega_0^3 + \omega_0^2)a_3 + \lambda\omega_0^4 a_4 + (\lambda\omega_0^5 + \omega_0^4)a_5 \pmod{a_0 \wedge a_1}.$$

Thus, the focus x must satisfy the following equations:

$$\begin{cases} \lambda\omega_0^2 \qquad\quad = 0, & \lambda\omega_0^4 \qquad\quad = 0, \\ \omega_0^2 \quad + \lambda\omega_0^3 = 0, & \omega_0^4 \quad + \lambda\omega_0^5 = 0. \end{cases} \tag{5.129}$$

This system is consistent if and only if

$$\begin{vmatrix} \lambda & 0 \\ 1 & \lambda \end{vmatrix}^2 = 0.$$

It follows that the value $\lambda = 0$ is a quadruple root of this equation. Thus, each line $a_0 \wedge a_1$ of the congruence K^0 has a real quadruple singular point $f = a_1$. Applying equations (5.121), it is easy to prove that when the straight line $a_0 \wedge a_1$ describes the congruence K^0, this focus describes the plane $\pi = a_1 \wedge a_3 \wedge a_5$. Hence K^0 is a *parabolic* line congruence. □

5.3.6 Smooth Lines in Projective Planes. On a projective plane $\mathbb{A}\mathbb{P}^2$, where $\mathbb{A}$ is one of the algebras $\mathbb{C}, \mathbb{C}^1$, and $\mathbb{C}^0$, consider a smooth point submanifold Γ of real dimension three. A line Γ of the plane $\mathbb{A}\mathbb{P}^2$ is called an $\mathbb{A}$-*smooth line* if at any of its points Y, the tangent space $T_Y(\Gamma)$ coincides with a straight line $U \subset \mathbb{A}\mathbb{P}^2$ corresponding to the projective plane passing through Y.

With an $\mathbb{A}$-smooth line Γ, we associate a family of projective frames $\{A_0, A_1, A_2\}$ in such a way that $A_0 = Y$ and A_1 lies on the tangent U to the line Γ at the point Y. Then on the line Γ, the first of equations (5.117) takes the form

$$dA_0 = A_0\Omega_0^0 + A_1\Omega_0^1. \tag{5.130}$$

It follows that $\mathbb{A}$-smooth lines on a plane $\mathbb{A}\mathbb{P}^2$ are defined by the equation

$$\Omega_0^2 = 0. \tag{5.131}$$

The 1-form Ω_0^1 in equation (5.130) defines a displacement of the point A_0 along the curve Γ. So this form is a basis form on Γ.

By equations (5.118), we have

$$\Omega_0^1 = \begin{pmatrix} \omega_0^2 & \omega_0^3 \\ \omega_1^2 & \omega_1^3, \end{pmatrix}, \quad \Omega_0^2 = \begin{pmatrix} \omega_0^4 & \omega_0^5 \\ \omega_1^4 & \omega_1^5 \end{pmatrix},$$

where ω_i^j are real 1-forms. For the algebras $\mathbb{C}, \mathbb{C}^1$, and $\mathbb{C}^0$, they are related, respectively, by equations (5.119), (5.120), and (5.121). As a result, on the line $\Gamma \subset \mathbb{A}\mathbb{P}^2$, the following differential equations will be satisfied:

$$\omega_0^4 = 0, \;\; \omega_0^5 = 0. \tag{5.132}$$

These equations are equivalent to equations (5.131).

Because Ω_0^1 is a basis form on the line $\Gamma \subset \mathbb{A}\mathbb{P}^2$, the real forms ω_0^2 and ω_0^3 are linearly independent. The families of straight lines in the space $\mathbb{R}\mathbb{P}^5$ corresponding to these lines depend on two parameters and form a real three-dimensional ruled variety $X^3 \subset \mathbb{R}\mathbb{P}^5$. The varieties X^3 belong to the congruences K, K^1, and K^0 if $\Gamma \subset \mathbb{C}\mathbb{P}^2, \Gamma \subset \mathbb{C}^1\mathbb{P}^2$, and $\Gamma \subset \mathbb{C}^0\mathbb{P}^2$, respectively.

Theorem 5.9. *The tangent subspace $T_x(X^3)$ to the ruled variety X^3, corresponding in the space $\mathbb{RP}^5$ to a smooth line in the planes $\mathbb{CP}^2, \mathbb{C}^1\mathbb{P}^2$, and $\mathbb{C}^0\mathbb{P}^2$, is fixed at all points of its rectilinear generator L, and the variety X^3 is a submanifold with a degenerate Gauss map of rank $r \leq 2$.*

Proof. Consider a rectilinear generator $L = a_0 \wedge a_1$ of the variety X^3. By (5.132), the differentials of the points a_0 and a_1 have the form

$$\begin{cases} da_0 = \omega_0^0 a_0 + \omega_0^1 a_1 + \omega_0^2 a_2 + \omega_0^3 a_3, \\ da_1 = \omega_0^1 a_0 + \omega_1^1 a_1 + \omega_1^2 a_2 + \omega_1^3 a_3. \end{cases} \tag{5.133}$$

It follows that at any point $x \in a_0 \wedge a_1$, the tangent subspace $T_x(X^3)$ belongs to a three-dimensional subspace $\mathbb{RP}^3 \subset \mathbb{RP}^5$ defined by the points a_0, a_1, a_2, and a_3. Thus, the subspace $T_x(X^3)$ remains fixed along the rectilinear generator $L = a_0 \wedge a_1$, and X^3 is a variety with a degenerate Gauss map of rank $r \leq 2$. □

5.3.7 Singular Points of Varieties Corresponding to Smooth Lines in the Projective Spaces over Two-Dimensional Algebras.

We prove the following theorem.

Theorem 5.10. *Consider three-dimensional varieties X^3 with degenerate Gauss maps of rank $r \leq 2$ in the space $\mathbb{RP}^5$ corresponding to smooth lines in the projective planes $\mathbb{CP}^2, \mathbb{C}^1\mathbb{P}^2$, and $\mathbb{C}^0\mathbb{P}^2$ over the algebras of complex, double, and dual numbers. For the algebra $\mathbb{C}$, such a variety does not have real singular points, for the algebra $\mathbb{C}^1$, such a variety is a join formed by the straight lines connecting the points of two plane curves that are in general position, and for the algebra $\mathbb{C}^0$, such a variety is a subfamily of the family of straight lines intersecting a plane curve. In all these cases, the general solution of the system defining a variety X^3 depends on two functions of one variable.*

Proof. A rectilinear generator $L = a_0 \wedge a_1$ of a variety X^3 of rank two bears two foci. Let us find these foci for the varieties X^3 corresponding to the lines Γ in the planes $\mathbb{CP}^2, \mathbb{C}^1\mathbb{P}^2$, and $\mathbb{C}^0\mathbb{P}^2$. We assume that these foci have the form $x = a_1 + \lambda a_0$.

If a line $\Gamma \subset \mathbb{C}^1\mathbb{P}^2$, then equations (5.127) and (5.133) are satisfied. They imply that

$$dx \equiv (\omega_0^3 + \lambda\omega_0^2)a_2 + (\omega_0^2 + \lambda\omega_0^3)a_3 \pmod{a_0 \wedge a_1},$$

and for the focus x, we have

$$\omega_0^3 + \lambda\omega_0^2 = 0, \ \ \omega_0^2 + \lambda\omega_0^3 = 0.$$

This system is consistent if and only if

$$\begin{vmatrix} 1 & \lambda \\ \lambda & 1 \end{vmatrix} = 0,$$

i.e., if $\lambda = \pm 1$. Thus, the foci of the straight line $a_0 \wedge a_1$ are the points $a_1 + a_0$ and $a_1 - a_0$. These points belong to the focal planes π_1 and π_2 of the congruence K^1 and describe lines Γ_1 and Γ_2. Such varieties X^3 were called *joins* (see Example 2.6 in Section 2.4). Because each of the lines Γ_1 and Γ_2 on the planes π_1 and π_2 is defined by means of one function of one variable, a variety X^3 depends on two functions of one variable. The same result could be obtained by applying the Cartan test (see the book [BCGGG 90] by Bryant, Chern, Gardner, Goldsmith, and Griffiths) to the system of equations (5.132).

If a line $\Gamma \subset \mathbb{CP}^2$, then we can prove that a rectilinear generator $L = a_0 \wedge a_1$ of the ruled variety X^3 corresponding to Γ bears two complex conjugate foci belonging to complex conjugate focal planes π_1 and $\pi_2 = \overline{\pi}_1$ of the congruence K. Hence *in the real space* $\mathbb{RP}^5$, *the variety* X^3 *does not have real singular points.*

In the complex plane π_1, the focus f_1 can describe an arbitrary differentiable line. But such a line is defined by means of two functions of one real variable. Therefore, in this case the variety X^3 also depends on two functions of one real variable.

Finally, consider a variety $X^3 \subset \mathbb{RP}^5$ corresponding to a line $\Gamma \subset \mathbb{C}^0\mathbb{P}^2$. Such a variety is defined in $\mathbb{RP}^5$ by differential equations (5.121) and (5.132). Using the same method, we can prove that a rectilinear generator $L = a_0 \wedge a_1$ of the ruled variety X^3 corresponding to Γ bears a double real focus $f = a_1$ belonging to the focal plane π of the congruence K^0 and describing in this plane an arbitrary line.

We prove that in this case a variety X^3 is also defined by two functions of one variable. But now in order to prove this, we apply the Cartan test.

Taking exterior derivatives of equations (5.132) and applying equations (5.121), we obtain the following exterior quadratic equations:

$$\omega_0^2 \wedge \omega_2^4 = 0, \;\; \omega_0^2 \wedge \omega_2^5 + \omega_0^3 \wedge \omega_2^4 = 0. \tag{5.134}$$

It follows from (5.134) that

$$\omega_2^4 = p\omega_0^2, \;\; \omega_2^5 = q\omega_0^2 + p\omega_0^3. \tag{5.135}$$

We apply the Cartan test to the system of equations (5.132), (5.134), and (5.135). In addition to the basis forms ω_0^2 and ω_0^3, equations (5.134) contain two more forms ω_2^4 and ω_2^5. Thus, we have $q = 2$. The number of independent equations in (5.134) is also 2, i.e., $s_1 = 2$. As a result, $s_2 = q - s_1 = 0$, and the Cartan number

$$Q = s_1 + 2s_2 = 2.$$

Equations (5.135) show that the number S of parameters on which the general two-dimensional integral element depends is also 2, $S = 2$. Because $Q = S$, the system of equations (5.132) is in involution, and its general solution depends on two functions of one variable. □

5.3.8 Curvature of Smooth Lines over Algebras. Differentiating equation (5.131) defining a smooth line Γ in the plane $\mathbb{A}\mathbb{P}^2$, where $\mathbb{A} = \mathbb{C}, \mathbb{C}^1, \mathbb{C}^0$, and applying Cartan's lemma, we obtain

$$\Omega_1^2 = R\Omega_0^1, \tag{5.136}$$

where $R \in \mathbb{A}$. The quantity R is called the *curvature* of the line $\Gamma \subset \mathbb{A}\mathbb{P}^2$.

For a line Γ in the plane $\mathbb{C}^1\mathbb{P}^2$, in formula (5.136) we have

$$\Omega_0^1 = \begin{pmatrix} \omega_0^2 & \omega_0^3 \\ \omega_0^3 & \omega_0^2 \end{pmatrix}, \quad \Omega_1^2 = \begin{pmatrix} \omega_2^4 & \omega_2^5 \\ \omega_2^5 & \omega_2^4 \end{pmatrix}, \quad R = \begin{pmatrix} p & q \\ q & p \end{pmatrix},$$

and $\det R = p^2 - q^2$. If $\operatorname{rank} R = 2$, then the quantity R is not a zero divisor, and the rank of the ruled variety X that corresponds in $\mathbb{R}\mathbb{P}^5$ to the line Γ, is also equal to two. If $\operatorname{rank} R = 1$, then R is a zero divisor, $R \neq 0$, and the rank of the variety X is equal to one. Finally, if $R = 0$, then a line Γ is a straight line in the plane $\mathbb{C}^1\mathbb{P}^2$, and the variety X corresponding to Γ in $\mathbb{R}\mathbb{P}^5$ is a subspace $\mathbb{R}\mathbb{P}^3$.

For a line Γ in the plane $\mathbb{C}\mathbb{P}^2$, in formula (5.136) we have

$$\Omega_0^1 = \begin{pmatrix} \omega_0^2 & \omega_0^3 \\ -\omega_0^3 & \omega_0^2 \end{pmatrix}, \quad \Omega_1^2 = \begin{pmatrix} \omega_2^4 & \omega_2^5 \\ -\omega_2^5 & \omega_2^4 \end{pmatrix}, \quad R = \begin{pmatrix} p & q \\ -q & p \end{pmatrix}.$$

Thus, $\det R = p^2 + q^2$, and two cases are possible: $\operatorname{rank} R = 2$ and $\operatorname{rank} R = 0$. In the first case, a variety $X \subset \mathbb{R}\mathbb{P}^5$ of rank two without singularities corresponds to the line $\Gamma \subset \mathbb{C}\mathbb{P}^2$, and in the second case, the line Γ is a straight line in the plane $\mathbb{C}\mathbb{P}^2$.

For a line Γ in the plane $\mathbb{C}^0\mathbb{P}^2$, in formula (5.136) we have

$$\Omega_0^1 = \begin{pmatrix} \omega_0^2 & \omega_0^3 \\ 0 & \omega_0^2 \end{pmatrix}, \quad \Omega_1^2 = \begin{pmatrix} \omega_2^4 & \omega_2^5 \\ 0 & \omega_2^4 \end{pmatrix}, \quad R = \begin{pmatrix} p & q \\ 0 & p \end{pmatrix},$$

and $\det R = p^2$. If $p \neq 0$, then $\operatorname{rank} R = 2$, and the curvature R is not a zero divisor. If $p = 0, q \neq 0$, then $\operatorname{rank} R = 1$, and the curvature R is a nonvanishing zero divisor. If $p = q = 0$, then $R = 0$. The rank of a variety X corresponding in $\mathbb{R}\mathbb{P}^5$ to a line $\Gamma \subset \mathbb{C}^0\mathbb{P}^2$ is equal to the rank of R. If $R = 0$, then the line Γ is a straight line in the plane $\mathbb{C}^0\mathbb{P}^2$.

Thus, we have proved the following result.

Theorem 5.11. *The rank of the ruled variety X corresponding in $\mathbb{R}\mathbb{P}^5$ to a smooth line $\Gamma \subset \mathbb{A}\mathbb{P}^2$, where $\mathbb{A} = \mathbb{C}, \mathbb{C}^1, \mathbb{C}^0$, is equal to the rank of the curvature of this line. For $A = \mathbb{C}$, this rank can be two or zero, and for $A = \mathbb{C}^1, \mathbb{C}^0$, the rank can be two, one, or zero.*

NOTES

5.1. The geometry of lightlike hypersurfaces on pseudo-Riemannian manifolds of different signatures was the subject of many journal papers and even two books: [DB 96] by Duggal and Bejancu and [Ku 96] by Kupeli.

On applications of the theory of lightlike hypersurfaces to physics see, for example, [Ch 83] by Chandrasekhar and [MTW 73] by Misner, Thorpe, and Wheeler.

Akivis and Goldberg in [AG 98b, 98c] studied the geometry of the de Sitter space $\mathbb{S}_1^{n+1}$ using its connection with the geometry of the conformal space. They proved that the geometry of lightlike hypersurfaces of the space $\mathbb{S}_1^{n+1}$ is directly connected with the geometry of hypersurfaces of the conformal space C^n. The latter was studied in detail in the papers of Akivis (see, for example, his paper [A 52]) and in the book [AG 96] by Akivis and Goldberg. This simplifies the study of lightlike hypersurfaces of the de Sitter space $\mathbb{S}_1^{n+1}$ and makes possible to apply for their consideration the apparatus constructed in the conformal theory.

In this section we follow the paper [AG 98c], namely, its parts in which the authors proved that a lightlike hypersurface has a degenerate Gauss map and where singular points of such hypersurfaces are investigated.

5.2. See more details on the geometry of normalized submanifolds and on construction of invariant intrinsic normalizations of submanifolds in the projective space $\mathbb{P}^n$ in the book Akivis and Goldberg [AG 93], Chapters 6 and 7.

On the normal connection see the paper [AG 95] and the book [AG 93] (Section 6.3) by Akivis and Goldberg.

On varieties with a flat normal connection in an affine space and a Euclidean space see the papers [ACh 75, 76, 01] by Akivis and Chakmazyan, where the authors indicated the ways to construct a classification of varieties with a flat normal connection.

Another relation of the theory of varieties with degenerate Gauss maps and the theory of normalized varieties was established in Theorem 4 of the paper [Cha 78] by Chakmazyan (see also p. 39 of his book [Cha 90]).

5.3. The theory of projective planes over algebras is the subject belonging to the geometry and the algebra, and this subject attracts the attention of both algebraists and geometers. This theory was considered in Pickert's book [Pi 75], and in the separate chapters of the books of Bourbaki (see [Bou 70]) and Rosenfeld (see [Ro 66, 97]).

However, not so much was known about the differential geometry of such projective planes. Some questions in this direction were considered in the paper [A 87b] by Akivis. In that paper the author studied smooth lines in projective planes over the matrix algebra and some of its subalgebras. In this study he used the mapping of the projective plane $\mathbb{MP}^2$ over the algebra $\mathbb{M}$ of $(n \times n)$-matrices onto the Grassmannian $\mathbb{G}(n-1, 3n-1)$ of subspaces of dimension $n-1$ of a real projective space $\mathbb{RP}^{3n-1}$.

It is proved in [A 87b] by Akivis that in the projective plane $\mathbb{MP}^2$ over the algebra $\mathbb{M}$ of (2×2)-matrices, there are no smooth lines different from straight lines. A family of straight lines in $\mathbb{RP}^5$ corresponding to those straight lines is the Grassmannian $\mathbb{G}(1,3)$ of straight lines lying in a three-dimensional subspace $\mathbb{RP}^3$ of the space $\mathbb{RP}^5$.

See more details on the plane $\mathbb{MP}^2$ and its mapping onto the Grassmannian $\mathbb{G}(n-1, 3n-1)$ of $(n-1)$-planes of a real projective space $\mathbb{P}^{3n-1}$ of dimension $3n-1$ in the paper [Ve 86] by Veselyaeva.

For description of the algebra $\mathbb{C}$ of complex numbers, the algebra $\mathbb{C}^1$ of double numbers, and the algebra $\mathbb{C}^0$ of dual numbers, see, for example, Paige [Pa 63], Schafer [Sc 66], or Rosenfeld [Ro 97].

In our exposition we follow the paper [AG 03a] by Akivis and Goldberg.

The examples we have constructed in this section are of the same nature as Ishikawa's examples in [I 99a], but they are much simpler.

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[1] In the bibliography we will use the following abbreviations for the review journals: JFM for *Jahrbuch für die Fortschritte der Mathematik*, MR for *Mathematical Reviews*, and Zbl. for *Zentralblatt für Mathematik und ihren Grenzgebiete*.

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SYMBOLS FREQUENTLY USED

The list below contains many of the symbols whose meaning is usually fixed throughout the book.

$\mathbb{A}^n$	affine space of dimension n, 25
$(\widetilde{a}^i_j)$	inverse matrix for a matrix (a^i_j), 3
(b^α_{ij})	second fundamental tensor of a variety, 54
C_x	cone with vertex at a point x, 58
$C(m,l)$	Segre cone, 44
C^n	conformal space of dimension n, 176
$\mathbb{C}^n$	n-dimensional complex space, 5
$\mathcal{C}$	correlation, 23
δ	symbol of differentiation with respect to secondary parameters, 7
δ_{ij}, δ^i_j	Kronecker symbol, 22, 27
$\delta_\gamma(X) = l$	Gauss defect (index of relative nullity) of X, 63, 71
$\delta_*(X)$	dual defect of X, 71, 72
$\mathbb{E}^n$	Euclidean space of dimension n, 26
F_L	focus hypersurface, 100
Φ_L	focus hypercone, 101
Φ, Φ_α	second fundamental form(s), 55
$\mathbf{GL}(n)$	general linear group, 1
$\mathbb{G}(n,N)$	Grassmannian of n-dimensional subspaces in $\mathbb{P}^N$, 41
$\gamma(X)$	Gauss map of X, 63
$\mathbb{H}^n$	hyperbolic space of dimension n, 126
H_∞	hyperplane at infinity, 118
L	l-dimensional generator of X, 64
L^n	vector space of dimension n, 1
$\Lambda^p(M)$	module of p-forms on M, 9
M, M^n	n-dimensional differentiable manifold, 5
$\widetilde{N}_x = T^{(2)}_x / T_x$	reduced normal subspace of X at a point x, 57
∇	differential operator, 4
∇_δ	operator of covariant differentiation relative to secondary parameters, 7
$\Omega(m,n)$	image of the Grassmannian $\mathbb{G}(m,n)$, 42
$\mathbf{PGL}(n)$	group of projective transformations, 20
$\widetilde{\mathbb{P}}^{n-m-1} = \mathbb{P}^n/\mathbb{P}^m$	projectivization of $\mathbb{P}^n$ with the center $\mathbb{P}^m$, 24
$\mathbb{P}^n$	projective space of dimension n, 19
$(\mathbb{P}^n)^*$	dual space of $\mathbb{P}^n$, 22

Q	Cartan's number, 14
$\mathcal{R}(M)$	frame bundle over M, 6
$\mathcal{R}^p(M)$	bundle of frames of order p over M, 17,
R^i_{jk}	the torsion tensor of an affine connection, 19
R^i_{jkl}	the curvature tensor of an affine connection, 19
$\mathbb{R}^n$	n-dimensional real space, 5
S	the arbitrariness of general integral element, 15
$\operatorname{Sing} X$	singular locus, 50
$\mathbf{SL}(n+1)$	special linear group, 21
$S(m,l)$	Segre variety, 44
$\mathbb{S}^n$	elliptic space of dimension n, 126
$s_1, s_2, \ldots$	characters, 14
$T(M^n)$	tangent bundle of M^n, 6
$T^*_x(M^n)$	dual tangent space of M^n at x, 6
$T^*(M^n)$	cotangent bundle of M^n, 6
$T_x(M^n)$	tangent space to M^n at a point x, 6
$T_x(X)$, T_x	tangent subspace to X at a point x, 51
$T^{(2)}_x(X)$	second osculating space to X at a point x, 56
$V(m)$	Veronese variety of dimension m, 45
$\mathbb{V}^n_c$	Riemannian manifold of dimension n and constant curvature c, 126
$\wedge$	symbol of exterior multiplication, 9
$X = V^n_r$	n-dimensional variety with degenerate Gauss map of rank r, 64
X_{sm}	locus of smooth points, 50

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